Comprehensive Examination, Fall 2001

REAL ANALYSIS

Instructions: Answer five out of the following eight questions. Please provide careful and complete answers; partial progress toward many questions counts less than a complete answer to fewer questions. If you attempt more than five questions, specify which ones you want to be graded, otherwise the first five answered will be graded.

1. Let $(X, \mathcal{A}, m)$ be a measure space with $m(X) < \infty$ and let $f : X \to X$ be measurable and satisfy

$$m(f^{-1}(E)) = m(E) \text{ for any measurable set } E.$$

Fix a measurable set $A \subset X$.

(i) Let $B = \{x \in A \mid f^n(x) \notin A \text{ for all } n > 0\}$. Show that $\{f^{-n}(B)\}_{n=0}^{\infty}$ are pairwise disjoint.

(ii) Show that $m(B) = 0$.

(iii) Show that for almost all $x \in A$, there are infinitely many $n > 0$ such that $f^n(x) \in A$.

2. Assume that $f, f_1, f_2, \ldots \in L^1(\mathbb{R})$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x.

(i) Show that $\Big\| |f_n| - |f_n - f| - |f| \Big\|_{L^1} \to 0$ as $n \to \infty$.

(ii) Show that if $\|f_n\|_{L^1} \to \|f\|_{L^1}$, then $\|f_n - f\|_{L^1} \to 0$.

(iii) Give an example of $\{f_n\} \subset L^1(\mathbb{R})$ such that

$$\lim_{n\to\infty} f_n(x) = 0 \text{ for all } x \in \mathbb{R}, \text{ but } \lim_{n\to\infty} \|f_n\|_{L^1} \neq 0.$$

3. Let f be a real-valued measurable function on a measure space $(X, \mathcal{A}, m)$. Define

$$g(\lambda) = m(\{x \in X \mid |f(x)| > \lambda\}) \qquad \lambda \geq 0.$$

(i) Show that $\displaystyle\int_X |f(x)| m(dx) = \int_0^\infty g(\lambda) d\lambda$.

(ii) Show that if f is integrable on X, then $\displaystyle\lim_{n\to\infty} ng(n) = \lim_{n\to\infty} \frac{1}{n} g\left(\frac{1}{n}\right) = 0$.

4. Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism satisfying

$$d(f(x), f(y)) \leq d(x, y) \qquad \text{for all } x, y \in X.$$

Show that:

(i) For any $a, b \in X$, $\lim_{n\to\infty} d(f^n(a), f^n(b))$ exists.

(ii) If $k_n \to \infty$ and $(f^{k_n}(a), f^{k_n}(b)) \to (x, y)$, then $d(f(x), f(y)) = d(x, y)$.

(iii) For any $(x, y) \in X \times X$, there exists a sequence $k_n \to \infty$ such that $(f^{-k_n}(x), f^{-k_n}(y))$ converges to some $(a, b) \in X \times X$.

(iv) Show that f indeed is an isometry; that is,

$$d(f(x), f(y)) = d(x, y) \qquad \text{for all } x, y \in X.$$

5. (a) Let H be a Hilbert space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous linear functionals on H such that for every $x \in H$

$$\sup_n |f_n(x)| < +\infty.$$

Show that

$$\sup_n \|f_n\| < +\infty.$$

Hint: Use the Baire Category Theorem.

(b) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that for every sequence $\{b_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} b_n^2 < \infty$, we have that $\sum_{n=1}^{\infty} a_n b_n$ is convergent. Prove that $\sum_{n=1}^{\infty} a_n^2 < \infty$.

6. Prove that a metric space X is complete if and only if for every nested sequence of non-empty closed sets A_n, $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$, such that $\text{diam}(A_n) \to 0$ we have

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

7. Let $g : [0,1] \to \mathbb{R}$ be continuous. Define

$$f(x) = \int_0^x g(t)dt.$$

Prove that f has bounded variation on $[0,1]$ and its total variation is equal to

$$\int_0^1 |g(x)|dx.$$

8. Let $\{f_n\}$ be a sequence of functions, each continuously differentiable on $(0,1)$, such that $\int_0^1 [f_n'(x)]^2 \, dx \leq 1$ and $\int_0^1 [f_n(x)]^2 \, dx \leq 1$ for every n. Prove that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges uniformly on $(0,1)$.

Hint: The Cauchy-Schwarz Inequality is helpful here.

Comprehensive Examination, Fall 2001

ALGEBRA

Instructions: Answer five out of the following eight questions. Please provide careful and complete answers; partial progress toward many questions counts less than a complete answer to fewer questions. If you attempt more than five questions, specify which ones you want to be graded, otherwise the first five answered will be graded.

1. Let G be a finite group.
   (a) Suppose that G has order n and p is the smallest prime divisor of n. Show that any subgroup of G of index p is normal.
   (b) Suppose that G has order 255. Prove that G has a normal subgroup of order 17 as well as a cyclic normal subgroup of order 85. From these facts deduce that any group of order 255 is cyclic.

2. Let $\gamma = (i_1, \ldots, i_r)$ be a cycle in S_n (the symmetric group on $\{1, \ldots, n\}$) and let σ be another element of S_n.
   (a) Show that $\sigma\gamma\sigma^{-1}$ is the cycle $(\sigma(i_1), \ldots, \sigma(i_r))$.
   (b) Let A_n be the subgroup of S_n consisting of the even permutations. Prove that A_n is generated by the 3-cycles and if $n \geq 5$ all 3-cycles are conjugates in A_n.

3. Let R be a commutative ring with unit element.
   (a) Prove that every maximal ideal of R is a prime ideal.
   (b) Give an example of a ring that is not a field in which every prime ideal is a maximal ideal.
   (c) Give an example of a ring in which some prime ideal is not a maximal ideal.

4. Let A be a *commutative* ring and I an ideal distinct from A. Let $J(I) = \{x \in A \mid x^n \in I \text{ for some } n > 0\}$.
   (a) Prove that $J(I)$ is an ideal and that $J(J(I)) = J(I)$.
   (b) Prove that $J(I)$ is equal to the intersection of the prime ideals of A containing I.

5. Let k be a field of characteristic $\neq 2, 3$. Prove that the following statements are equivalent:
   (a) Any sum of squares in k is itself a square.
   (b) Whenever a cubic polynomial f factors completely in k, so does its derivative f'.

6. (a) Let K be a field and F a subfield of K. Let $a, b \in K$ be algebraic over F of degree m, n, respectively. Prove that $a + b$ is algebraic over F of degree at most mn.
   (b) If m, n are relatively prime, show that $F(a, b)$ is an algebraic extension of degree exactly mn over F.
   (c) Let $K = \mathbb{R}$ be the field of real numbers and $F = \mathbb{Q}$ the subfield of rational numbers. What is the degree of $\sqrt{2} + \sqrt{3}$ over F? You have to prove your answers in order to get credit.

7. Let V be a finite dimensional real vector space, and $T : V \to V$ be a linear transformation. Prove that $\mathrm{Rank}(T) = \mathrm{Rank}(T^2)$ if and only if

$$\mathrm{Range}(T) \cap \mathrm{Nullspace}(T) = \{0\}.$$

8. Prove that for every $n \times n$ complex matrix A there exists a unitary matrix U such that $M = U^* AU$ is upper triangular. (U is unitary if $A^* = A^{-1}$, where $A^* = \bar{A}^T$.)