Instructions: Attempt any **five** questions, and please provide **careful and complete answers with proofs**. If you attempt more questions, specify which five should be graded. Otherwise, by default, only the first five will be graded.

1. **(a)** For $1 < p < \infty$, show that for each $f \in L^p([0,1], \mathrm{d}x)$ there is a *unique* $g \in L^q([0,1], \mathrm{d}x)$, where $1/p + 1/q = 1$ so that

$$\int_{[0,1]} fg\mathrm{d}x = \|f\|_p \qquad \text{and} \qquad \|g\|_q = 1 \ . \tag{$*$}$$

(b) Give an example of an $f \in L^1([0,1], \mathrm{d}x)$ for which there are *infinitely many* $g \in L^\infty([0,1], \mathrm{d}x)$ so that $(*)$ holds.

(c) Give an example of an $f \in L^\infty([0,1], \mathrm{d}x)$ for which there is *no* $g \in L^1([0,1], \mathrm{d}x)$ so that $(*)$ holds.

SOLUTION By Hölders inequality, and then the hypothesis that $\|g\|_q = 1$,

$$\left|\int_{[0,1]} fg\mathrm{d}x\right| \le \|f\|_p\|g\|_q = \|f\|_p \ .$$

Hence for $(*)$ to hold, g must be such that there is equality in Hölder's inequality. For $p < \infty$, this is the case if and only if g is a constant multiple of $|f(x)|^{p-2}f^*(x)$. (Those who do not remember the exact condition for the cases of equality can easily derive them if they remember the proof in terms of the arithmetic–geometric mean inequality.)

It is now easy to answer the questions.

(a) For $1 < p < \infty$, let

$$g(x) = \frac{|f(x)|^{p-2}f^*(x)}{\|f\|_p^{p-1}} \ . \tag{$**$}$$

By what we have noted above concerning case of equality in Hölder's inequality, if there is any such g, this must be it Let's check that it works.

Since $q = p/(p-1)$, $|g(x)|^q = |f(x)|^p/\|f\|_p^p$ so that $\int_{[0,1]} |g(x)|^q\mathrm{d}x = \|f\|_p^p/\|f\|_p^p = 1$. That, $\|g\|_q = 1$. Also

$$\int_{[0,1]} fg\mathrm{d}x = \int_{[0,1]} |f|^p/\|f\|_p^{p-1}\mathrm{d}x = \|f\|_p \ .$$

So this works and hence $(**)$ gives the unique element of $L^q([0,1], \mathrm{d}x)$ for which $(*)$ holds.

(b) Suppose that $f(x) = 2$ on $[0,1/2]$ and $f(x) = 0$ elsewhere. For any number a with $|a| \le 1$, let $g(x) = 2$ on $[0,1/2]$ and $g(x) = a$ elsewhere.

Then $\int_{[0,1]} fg\mathrm{d}x = 1 = \|f\|_1$ while $\|g\|_\infty = 1$. Since there are infinitely many a with $|a| \le 1$, there are infinitely many such functions g.

(c) Let $f(x) = x$. Then $\|f\|_\infty = 1$. Let g be any function in $L^1([0,1])$ with $\|g\|_1 = 1$. Then by dominated convergence, there is some n so that

$$\int_{[0,1-1/n]} |g(x)|\mathrm{d}x \geq 1/2 \ .$$

But then

$$\begin{aligned}
\left|\int_{[0,1]} fg\mathrm{d}x\right| &\leq \int_{[0,1-1/n]} |fg|\mathrm{d}x + \int_{[1-1/n,1]} |fg|\mathrm{d}x \\
&\leq (1-1/n)\int_{[0,1-1/n]} |g|\mathrm{d}x + \int_{[1-1/n,1]} |g|\mathrm{d}x \\
&= |\int_{[0,1]} |g|\mathrm{d}x - \frac{1}{n}\int_{[0,1-1/n]} |g|\mathrm{d}x \\
&= |\int_{[0,1]} |g|\mathrm{d}x - \frac{1}{n}\int_{[0,1-1/n]} |g|\mathrm{d}x \\
&\leq 1 - 1/2n < 1 \ .
\end{aligned}$$

So there is no such g for this f.

2. Is there a function $f \in L^p([0,1], \mathrm{d}x)$ for all $1 \le p < \infty$ such that for each x in $[0,1]$,

$$\limsup_{z \to x} f(z) = +\infty \qquad \text{and} \qquad \liminf_{z \to x} f(z) = -\infty \ ?$$

Either prove that there is no such function, or give an example.

SOLUTION There are such functions. We will first explain something quite standard. You can skip ahead to the **full solution** below, which is self contained. But anyone who did the standard thing had some key ideas and got substantial credit, so we explain that first.

Let's first show that there is a function $f \in L^p([0,1], \mathrm{d}x)$ for all $1 \le p < \infty$ such that for each x in $[0,1]$,

$$\limsup_{z \to x} f(z) = -\infty \ .$$

For $x \in R$, let $\phi(x) = \ln(|x|)$ for $|x| \le 1$, and 0 otherwise. This is in L^p for each $p < \infty$, by comparison with a small negative power of $|x|$, but $\phi(0) = \infty$. Next, for any number sequence $\{a_n\}$ of positive numbers a_n, let $\phi_n(x) = \phi(a_n x))$. Then

$$\|\phi_n\|_p = |a_n|^{-1/p} \|\phi\|_p \ ,$$

and $\phi_n(0) = \infty$. Now let $\{q_n\}$ be some enumeration of the rational numbers in $[0,1]$, and specify the sequence $\{a_n\}$ by $a_n = 2^n$. Define

$$f(x) = \sum_{n=1}^{\infty} \phi_n(x - q_n) \ ,$$

restricted to $[0,1]$. Then by Minkowskii's inequality,

$$\|f\|_p \le \sum_{n=1}^{\infty} 2^{-n/p} \|\phi\|_p$$

which is finite for all $p < \infty$. Thus, $f \in L^p([0,1], \mathrm{d}x)$ for all $p < \infty$. Also clearly for any given number L, at each rational number q_n, $f(x) < -L$ on some interval about q (with a width depending on q), just because this is true of $\phi_n(x - a_n)$. Since each interval around each x in $[0,1]$ contains rational numbers, there will be a set of positive measure on which $f \le -L$ in every interval about x. Therefore (independently of the representative of the L^p equivalence class)

$$\limsup_{z \to x} f(z) = -\infty \ .$$

Full solution What we just did is standard, but anyone who got this got pretty substantial credit. To go further, a natural first try would be

$$\sum_{n=1}^{\infty} (-1)^n \phi_n(x - q_n) \ .$$

But now that there are different signs, one has to deal with possible cancelation. It is easier to modify the construction slightly to facilitate this.

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers. Let $\psi_n(x) = a_n$ for $|x| < b_n$ 0 otherwise. Let

$$\sum_{n=1}^{\infty}(-1)^n \psi_n(x - q_n) \ .$$

We will choose the sequence $\{b_n\}$ so that for each N,

$$b_N > 2\left(\sum_{m=N+1}^{\infty} b_m\right) \ . \tag{1}$$

In this case, the union of supports of the $\psi_m(x - q_m)$ for $m > N$ is less than half the support of $\psi_N(x - q_N)$, and so there cannot be too much cancellation from later terms:

$$\left|\sum_{m=N}^{\infty} \psi_m(x - q_m)\right| \geq a_N$$

on a set of measure at least b_N. Next, we choose the sequence $\{a_n\}$ so that

$$a_N > 2\left(\sum_{m=1}^{N-1} a_m\right) \ . \tag{2}$$

In this case there cannot be too much cancelation from the first $N - 1$ terms, and so

$$(-1)^N \left(\sum_{m=1}^{\infty} \psi_m(x - q_m)\right) \geq a_N/2 \tag{3}$$

on a set of measure at least b_N.

Next, $\|\phi_n\|_p = a_n(2b_n)^{1/p}$, so we also need the sequences $\{a_n\}$ and $\{b_n\}$ to satisify

$$\sum_{n=1}^{\infty} a_n(2b_n)^{1/p} < \infty \tag{4}$$

for all p with $1 < p < \infty$. If (3) is satisfied, then $f \in L^p$ by Minkowskii's inequality. And by (3), for every x, and every $\epsilon > 0$, for all N large enough, there is a subset of $[x - \epsilon, x + \epsilon]$ of positive measure in which $(-1)^N f \geq a_N/2$, so as long as this sequence ends to infinity, we have the result.

It remains to show that there are sequences $\{a_n\}$ and $\{b_n\}$ satisfying (1), (2) and (4). This is easy. Fix any number $R > 1$. Define

$$a_n = R^{2^n} \qquad \text{and} \qquad b_n = R^{-(n^n)} \ .$$

It is pretty clear from familiar facts about geometric sums being close in value to the mist significant term that these work. But since questions were asked, here are the gory details:

To see that this works, note

$$\sum_{m=N+1}^{\infty} b_m \le \sum_{k=(N+1)^{N+1}}^{\infty} R^{-k} = R^{-(N+1)^{N+1}} \frac{1}{1-R^{-1}}$$

since the sum is over distinct powers of R^{-1}, the smallest of which is $(N+1)^{N+1}$. Therefore,

$$\frac{\sum_{m=N+1}^{\infty} b_m}{b_N} \le \frac{R^{-(N+1)^{N+1}}}{R^{-N^N}} \frac{1}{1-R^{-1}} \le R^{-5N/4} \frac{1}{1-R^{-1}} \ .$$

Clearly, we can easily choose R so that (1) is satisfied.

Likewise,

$$\sum_{m=1}^{N-1} a_m \le \sum_{m=1}^{2^{N-1}} R^m = \frac{R^{2^{N-1}+1}-1}{R-1}$$

so that

$$\frac{\sum_{m=1}^{N-1} a_m}{a_N} \le \frac{R^{2^{N-1}+1}-1}{R^{2^N}} \frac{1}{R-1} \le R^{-2^{N-1}} \frac{1}{R-1}$$

Clearly, we can easily choose R so that (2) is satisfied.

Finally, consider

$$\sum_{n=1}^{\infty} a_n (2b_n)^{1/p} = \sum_{n=1}^{\infty} R^{2^n - n^n/p} \ .$$

Notice that

$$2^n - n^n/p = -2^n((n/2)^n/p - 1)$$

and $(n/2)^n/p > 2$ for all n large enough, so

$$2^n - n^n/p \le -2^n < -n$$

for all n large enough, so the series converges by comparison with the geometric series.

3. **(a)** Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 < p < \infty$, and suppose that f is a measurable function on X such that for some $C < \infty$

$$\int_A |f(x)| \mathrm{d}\mu \le C\mu(A)^{1/p'} \tag{$*$}$$

for every measurable set $A \subset X$, where $1/p + 1/p' = 1$. Show that this does not imply that $f \in L^p(X, \mathcal{S}, \mu)$.

(b) Suppose in addition to $(*)$ that for some q with $p < q < \infty$, there is a constant $D < \infty$ such that

$$\int_A |f(x)| \mathrm{d}\mu \le D\mu(A)^{1/q'} \tag{$**$}$$

for every measurable set $A \subset X$, where $1/q + 1/q' = 1$. Show that then $f \in L^r(X, \mathcal{S}, \mu)$ for each r with $p < r < q$.

SOLUTION (a) Let $X = R_+$, and let μ be Lebesgue measure. Let $f(x) = 1/x^{1/p}$. Then if $\mu(A) = a$, it is clear that

$$\int_A |f(x)| \mathrm{d}\mu \le \int_{[0,a]} |f(x)| \mathrm{d}\mu = \int_{[0,a]} x^{-1/p} \mathrm{d}x = \frac{1}{p'} a^{-1/p'} \ .$$

Thus, $(*)$ holds with $C = 1/p'$. However, f is not in L^p.

(b) For each $t > 0$, let $h(t) = \mu(\{x \ : \ |f(x)| > t\ \})$. Then for any $1 < r < \infty$,

$$\|f\|_r^r = -\int_0^\infty t^p \mathrm{d}h(t) = r\int_0^\infty t^{r-1} h(t) \mathrm{d}t \ . \tag{$**$}$$

We can use $(**)$ to estimate L^r norms if we can estimate $h(t)$. We can do this using $(*)$ and $(**)$ if we consider the set $A = \{x \ : \ |f(x)| > t\ \}$. By $(*)$,

$$th(t) \le \int_{\{x \ : \ |f(x)| > t\ \}} |f(x)| \mathrm{d}\mu \le Ch(t)^{1-1/p} \ .$$

That is,

$$h(t) \le (C/t)^p \ .$$

Likewise, form $(**)$ we deduce

$$h(t) \le (C/t)^q \ .$$

We then have

$$\begin{aligned} r\int_0^\infty t^{r-1} h(t) \mathrm{d}t &\le r\int_0^1 t^{r-1} (C/t)^p \mathrm{d}t + r\int_1^\infty t^{r-1} (D/t)^r \mathrm{d}t \\ &\le rC^p \int_0^1 t^{r-1-p} \mathrm{d}t + \le rD^q \int_1^\infty t^{r-1-q} \mathrm{d}t \ . \end{aligned}$$

Since $p < r < q$, both of these integral converge, and then by $(**)$, $f \in L^r$.

4 Let $F(x,y)$ be a continuous function on $[0,1]\times[0,1]$. Define a linear transformation $T:\mathcal{C}([0,1])\to\mathcal{C}([0,1])$ by

$$Tf(x)=\int_0^1 F(x,y)f(y)\mathrm{d}y\ .$$

Show that if $\{f_n\}$ is any sequence in $\mathcal{C}([0,1])$ with

$$\sup_n \|f_n\|_{\mathcal{C}([0,1])}<\infty\ ,$$

then there is a subsequence of $\{Tf_n\}$ that is strongly convergent in $\mathcal{C}([0,1])$.

SOLUTION Use Arzelà–Ascoli to get compactness...

5 Let S be a closed linear subspace of $L^1([0,1])$ with the property that for each individual $f \in S$, there is some $p > 1$ so that $f \in L^p([0,1])$. Show that there is then some $p > 1$ so that $S \subset L^p([0,1])$.

SOLUTION Let $S_n = S \cap L^{1+1/n}([0,1])$. Show that these sets are closed in S using Fatou, pointwise convergent subsequence... Then apply Baire's Theorem.

6. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f \in L^1(X, \mu)$. Show that there exists a convex increasing function $\phi : [0, \infty) \to \mathbf{R}$ such that

$$\phi(0) = 0, \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty,$$

and

$$\phi(|f|) \in L^1(X, \mu).$$

SOLUTION Recall that for any measurable function f on $(X, \mathcal{S}, \mu)$,

$$\int_X |f| d\mu = \int_0^\infty \omega(\lambda) d\lambda,$$

where

$$\omega(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right) \qquad \text{for } \lambda \in [0, \infty).$$

This can be proved using Fubini's theorem. In particular, $f \in L^1(X, \mu)$ if and only if $\omega(\lambda)$ is in $L^1[0, \infty)$.

We first consider the trivial case where $|f| \in L^\infty(X, \mu)$. Denote $M = \|f\|_{L^\infty} < \infty$. Define

$$\phi(t) = t \quad (0 \le t \le M), \qquad \phi(t) = t + (t - M)^2 \quad (t > M).$$

Then ϕ satisfies the requirements.

Next consider the case where $|f| \notin L^\infty(X, \mu)$. Let

$$\rho(\lambda) = \int_\lambda^\infty \omega(t) dt \qquad t \ge 0.$$

We see that $\rho(0) = \|f\|_{L^1} < \infty$, $\rho(\lambda)$ is positive and decreasing, $\rho(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$, $\rho(\lambda)$ is absolutely continuous, and $\rho'(\lambda) = -\omega(\lambda)$ a.e. $\lambda \in (0, \infty)$. Define

$$\phi(0) = 0, \qquad \phi(t) = \int_0^t \rho(\lambda)^{-1/2} d\lambda \qquad t \ge 0.$$

Since $\phi'(t) = \rho(t)^{-1/2} > 0$ is increasing, it follows that $\phi(t)$ is convex and strictly increasing. It is also easy to see

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \rho(\lambda)^{-1/2} d\lambda = \infty,$$

since $\rho(\lambda)^{-1/2} \to \infty$ as $\lambda \to \infty$. Finally, let us verify $\phi(|f|) \in L^1(X, \mu)$. Notice that

$$\mu\left(\{x \in X : \phi(|f(x)|) > \lambda\}\right) = \mu\left(\{x \in X : |f(x)| > \phi^{-1}(\lambda)\}\right) = \omega(\phi^{-1}(\lambda)).$$

Hence,

$$\begin{aligned}
\int_X \phi(|f|)d\mu &= \int_0^\infty \omega(\phi^{-1}(\lambda))d\lambda \\
&= \int_0^\infty \omega(t)\phi'(t)dt \qquad \text{(change of variable } \lambda = \phi(t)\text{)} \\
&= \int_0^\infty \omega(t)\rho(t)^{-1/2}dt \\
&= \int_0^\infty -\rho'(t)\rho(t)^{-1/2}dt \\
&= \Big[-2\rho(t)^{1/2}\Big]_0^\infty \qquad \text{(since } \rho(t) \text{ is abs. conti.)} \\
&= 2\rho(0)^{1/2} = 2\|f\|_{L^1}^{1/2} < \infty.
\end{aligned}$$

7. Let $f : [0,1] \to \mathbf{R}$ be continuous, $g : [0,1] \to \mathbf{R}$ Lebesgue measurable, and $0 \le g(x) \le 1$ for a.e. $x \in [0,1]$. Find the limit:

$$\lim_{n\to\infty} \int_0^1 f\left(g(x)^n\right) dx.$$

SOLUTION Define

$$A = \{x \in [0,1] : g(x) = 1\}, \quad B = \{x \in [0,1] : 0 \le g(x) < 1\}.$$

By the assumption, $A \cup B$ is of full measure in $[0,1]$; that is, $\mu(A) + \mu(B) = 1$.

For every $x \in A$, $f\left(g(x)^n\right) = 1$.

For every $x \in B$, $g(x)^n \to 0$ as $n \to \infty$. Combining this with the continuity of f, we obtain $f\left(g(x)^n\right) \to f(0)$.

Since f is continuous on a compact set $[0,1]$, $|f|$ is bounded: $|f(t)| \le M < \infty$ for all $x \in [0,1]$. This implies the boundedness of the integrand:

$$\left|f\left(g(x)^n\right)\right| \le M \qquad \text{for all } x \in A \cup B.$$

By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 f\left(g(x)^n\right) dx = \int_A f(1) dx + \int_B f(0) dx = f(1)\mu(A) + f(0)[1 - \mu(A)].$$

8. Let X and Y be metric spaces and $f : X \to Y$ be a mapping. Show that if the restriction of f on any compact subset of X is continuous, then f is continuous on X.

SOLUTION Let $x_n \to x$ in X. We need to show $f(x_n) \to f(x)$. Define

$$K = \{x_1, x_2, \cdots\} \cup \{x\}.$$

It is easily seen that K is a compact subset of X. By the assumption, f is continuous on K. This implies $f(x_n) \to f(x)$.