Analysis Comprehensive Exam Questions
Fall 2011

1. Let $f \in L^2(0,\infty)$ be given.

(a) Prove that
$$\left(\int_0^x f(t)\,dt\right)^2 \le 2\sqrt{x}\int_0^x \sqrt{t}\,f(t)^2\,dt.$$

(b) Given part (a), prove that $\|F\|_{L^2(0,\infty)} \le 2\,\|f\|_{L^2(0,\infty)}$, where
$$F(x) = \frac{1}{x}\int_0^x f(t)dt.$$

Solution

(a) Using Hölder's inequality we have
$$\left(\int_0^x f(t)dt\right)^2 = \left(\int_0^x \frac{1}{t^{\frac{1}{4}}}t^{\frac{1}{4}}f(t)dt\right)^2 \le \int_0^x \frac{1}{t^{\frac{1}{2}}}dt\int_0^x t^{\frac{1}{2}}f^2(t)dt = 2\sqrt{x}\int_0^x \sqrt{t}f(t)^2dt.$$

(b) By (a)
$$F^2(x) \le \frac{2}{\sqrt{x}^3}\int_0^x \sqrt{t}f^2(t)dt.$$

Set
$$D = \{(t,x)\mathbb{R}^2 \mid 0 < t \le x < \infty\} \quad \text{and} \quad g(t,x) = \frac{2\sqrt{t}}{\sqrt{x}^3}\chi_D(t,x)f^2(t).$$

We conclude that
$$||F||^2_{L^2(0,\infty)} \le \int_0^\infty \big(\int_0^x \frac{2\sqrt{t}}{\sqrt{x}^3}f^2(t)dt\big)dx = \int_{\mathbb{R}^2} g(t,x)dtdx.$$

As g is nonnegative we may apply Tonelli's theorem to obtain that
$$\begin{aligned}||F||^2_{L^2(0,\infty)} &\le \int_{\mathbb{R}^2} g(t,x)dxdt \\ &= \int_0^\infty 2\sqrt{t}f^2(t)\big(\int_t^\infty \frac{1}{\sqrt{x}^3}dx\big)dt \\ &= \int_0^\infty 2\sqrt{t}f^2(t)dt\frac{2}{\sqrt{t}}dx \\ &= 4||f||^2_{L^2(0,\infty)}.\end{aligned}$$

2. Let X be a compact metric space, and let $\cdots \subset X_n \subset X_{n-1} \subset \cdots X_2 \subset X_1$ be a nested sequence of closed nonempty subsets of X.

(a) Show that $S := \cap_{n=1}^{\infty} X_n$ is nonempty.

(b) Suppose that none of the X_n is contained in the disjoint union of two nonempty open sets. Show that S cannot be contained in the disjoint union of two nonempty open sets (hence S is connected).

Solution

(a) Denote by X_n^c the complement of X_n. I if Y was empty, $\{X_n^c\}_{n=1}^{\infty}$ would be an open cover of X. As X is compact, we may a finite subcover $\{X_{n_i}\}_{i=1}^{k}$ where $n_1 < n_2 \cdots, n_k$ are positive integers. This would imply that such that $\emptyset = \cap_{i=1}^{k} X_{n_i} = X_{n_k} \neq \emptyset$ which yields a contradiction.

(b) Suppose on the contrary that $S \subset U \cup V$ where U and V are open, non empty and disjoint. Then the sets $F_n = X_n \setminus (U \cup V)$ are closed and $\{F_n\}_{n=1}^{\infty}$ is a nested family whose intersection is empty. Thus, by (a) there exists n_0 such that F_{n_0} is empty. This yields a contradiction.

3. Let $U\colon H \to H$ be a unitary mapping on a Hilbert space H. Let I be the identity map on H, set $M = \ker(U - I)$, let P be the orthogonal projection of H onto M, and for each integer $N > 0$ define

$$S_N = \frac{1}{N} \sum_{k=0}^{N-1} U^k,$$

where we take $U^0 = I$. Show that $S_N \to P$ strongly, i.e., for each $f \in H$ we have $S_N f \to Pf$ as $N \to \infty$.

Hint: Show that $M^\perp = \overline{\mathrm{range}(U - I)}$. Consider $f \in M$ and $f \in \mathrm{range}(U - I)$ first.

<u>Solution</u>

We have

$$Uf - f = 0 \qquad \Longleftrightarrow \qquad f - U^{-1}f = 0,$$

so

$$\ker(U - I) = \ker(U^{-1} - I) = \ker(U^* - I).$$

Therefore

$$M^\perp = \ker(U - I)^\perp = \ker(U^* - I)^\perp = \overline{\mathrm{range}((U^* - I)^*)} = \overline{\mathrm{range}(U - I)}.$$

If $f \in M = \ker(U-I)$, then $Uf = f$ and therefore $S_N f = f$ for every N. Hence $S_N f \to Pf$ in this case.

If $f \in \mathrm{range}(U - I)$, then $f = Ug - g$ for some $g \in H$. Therefore

$$S_N f = \frac{1}{N} \sum_{k=0}^{N-1} U^k f = \frac{1}{N} \sum_{k=0}^{N-1} (U^{k+1}g - U^k g) = \frac{U^N g}{N} - \frac{g}{N}.$$

Hence

$$\|S_N f\| \le \frac{\|U^N g\|}{N} + \frac{\|g\|}{N} = \frac{2\,\|g\|}{N} \to 0.$$

Suppose $f \in M^\perp = \overline{\mathrm{range}(U - I)}$, and fix $\varepsilon > 0$. Then there exists some $g \in \mathrm{range}(U - I)$ such that $\|f - g\| < \varepsilon$. Hence

$$\begin{aligned}
\|S_N f\| &\le \|S_N g\| + \|S_N(f - g)\| \\
&\le \|S_N g\| + \frac{1}{N} \sum_{k=0}^{N-1} \|U^k(f - g)\| \\
&= \|S_N g\| + \|f - g\| \\
&\le \|S_N g\| + \varepsilon.
\end{aligned}$$

Consequently

$$\limsup_{N\to\infty} \|S_N f\| \le \limsup_{N\to\infty} \|S_N g\| + \varepsilon = \varepsilon.$$

Since ε is arbitrary, it follows that $S_N f \to 0$ as $N \to \infty$.

Finally, given an arbitrary vector $f \in H$, we have $f = g + h$ where $g = Pf \in M$ and $h \in M^\perp$. Therefore, by combining the cases above we see that

$$S_N f = S_N g + S_N h \to g + 0 = Pf. \qquad \square$$

4. Let X be a complete metric space. Prove that if $U_1, U_2, \dots$ are open, dense subsets of X, then $\cap U_n$ is dense in X.

Solution

The sets $E_n = U_n^{\mathrm{C}} = X \setminus U_n$ are closed and have empty interiors.

If $\cap U_n$ is not dense in X, then we can find some $f \in X$ and $r > 0$ such that

$$B_r(f) \subseteq \Big(\bigcap_{n=1}^{\infty} U_n\Big)^{\mathrm{C}} = \bigcup_{n=1}^{\infty} E_n.$$

The closed ball

$$Y = \overline{B_{r/2}(f)}$$

is a closed subset of X and hence is a complete metric space (using the metric on X). We have

$$Y = \bigcup_{n=1}^{\infty} (E_n \cap Y).$$

Each set $E_n \cap Y$ is closed in Y, and its complement in Y is

$$Y \setminus (E_n \cap Y) = Y \cap U_n.$$

Since U_n is dense in X, the set $Y \cap U_n$ is dense in Y. Hence the interior of $E_n \cap Y$ must be empty. Therefore we have written Y as a countable union of closed sets that each have empty interiors. The Baire Category Theorem says that a complete metric space is nonmeager in itself, so this is a contradiction.

5. Given $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, define $f^y(x) = f(x - y)$.

(a) Show that if f is continuous and has compact support, then $\lim_{y\to 0} \|f^y - f\|_{L^\infty(\mathbb{R})} = 0$.

(b) Show that if $f \in L^p(\mathbb{R})$ and $1 \le p < \infty$, then $\lim_{y\to 0} \|f^y - f\|_{L^p(\mathbb{R})} = 0$.

(c) Either prove or give a counterexample: if $f \in L^\infty(\mathbb{R})$, then $\lim_{y\to 0} \|f^y - f\|_{L^\infty(\mathbb{R})} = 0$.

Solution

(a) Suppose f is continuous its support is contained in $[-r, r]$ and $r > 1$. Note that if $|y| < 1$ then $f^y(x) - f(x) = f(x - y) - f(x) = 0$ unless $|x| \le 2r$. Hence, if $\epsilon \in (0, 1)$

$$l(\epsilon) := \sup_{|y|\le\epsilon} ||f^y - f||_{L^\infty(\mathbb{R})} = \sup_{x,y}\{|f(x - y) - f(x)| \mid x \in [-2r, 2r], |y| \le \epsilon\}.$$

But, f is uniformly continuous on $[-4r, 4r]$ and so,

$$\lim_{\epsilon\to 0^+} \sup_{x,y}\{|f(x - y) - f(x)| \mid x \in [-2r, 2r], |y| \le \epsilon\} = 0.$$

This proves that $\lim_{\epsilon\to 0^+} \sup_{|y|\le\epsilon} ||f^y - f||_{L^\infty(\mathbb{R})}$ which proves (a).

(b) Suppose $f \in L^p(\mathbb{R})$. For each integer $n \ge 1$ we may find $g_n : \mathbb{R} \to \mathbb{R}$ continuous with compact support such that $||f - g_n||_{L^p(\mathbb{R})} < 1/n$. We use the triangle inequality to obtain that

$$||f^y - f||_{L^p(\mathbb{R})} \le ||f^y - g_n^y||_{L^p(\mathbb{R})} + ||g_n^y - g_n||_{L^p(\mathbb{R})} + ||g_n - f||_{L^p(\mathbb{R})}.$$

A simple change of variables reveal that $||f^y - g_n^y||_{L^p(\mathbb{R})} = ||g_n - f||_{L^p(\mathbb{R})} < 1/n$ and so,

$$||f^y - f||_{L^p(\mathbb{R})} \le ||g_n^y - g_n||_{L^p(\mathbb{R})} + \frac{2}{n}. \tag{1}$$

Assume that g_n is supported by $[-r_n + 1, r_n - 1]$ and $|y| \le 1$. Then $g_n^y - g_n$ is supported by $[-r_n, r_n]$ and so,

$$||g_n^y - g_n||_{L^p(\mathbb{R})} = ||g_n^y - g_n||_{L^p([-r_n,r_n])} \le (2r_n)^{\frac{1}{p}} ||g_n^y - g_n||_{L^\infty(\mathbb{R})}.$$

This, together with (a) implies

$$\limsup_{y\to 0} ||g_n^y - g_n||_{L^p(\mathbb{R})} \le \limsup_{y\to 0} (2r_n)^{\frac{1}{p}} ||g_n^y - g_n||_{L^\infty(\mathbb{R})} = 0. \tag{2}$$

We combine (1) and (2) to conclude that

$$\limsup_{y\to 0} ||f^y - f||_{L^p(\mathbb{R})} \le 2/n.$$

Since n is arbitrary we have that $0 \le \limsup_{y\to 0} ||f^y - f||_{L^p(\mathbb{R})} \le 0$ and so, $\lim_{y\to 0} ||f^y - f||_{L^p(\mathbb{R})} = 0$.

(c) Let $f = \chi_{(0,1)}$. In other words, $f(x) = 1$ if $x \in (0, 1)$ and $f(x) = 0$ if $x \notin (0, 1)$. For $0 < y < 1$ and $x \in (0, y)$, $f^y(x) - f(x) = -1$. Hence,

$$||f^y - f||_{L^\infty(\mathbb{R})} \ge ||f^y - f||_{L^\infty(0,y)} = 1.$$

This proves that we don't have $\lim_{y\to 0} ||f^y - f||_{L^\infty(\mathbb{R})} = 0$.

6. Assume f is absolutely continuous on an interval $[a, b]$, and there is a continuous function g such that $f' = g$ a.e. Show that f is differentiable at all points of $[a, b]$, and $f'(x) = g(x)$ for all $x \in [a, b]$.

Solution

First proof. Since g is continuous, every point is a Lebesgue point of g. Suppose that $x \in (a, b)$. If $|h|$ is small enough, then we have $x + h \in (a, b)$ as well, so we can compute that

$$\begin{aligned} g(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} g(t)\, dt && \text{Fund. Thm. Calculus} \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f'(t)\, dt && \text{since } f' = g \text{ a.e.} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{since } f \text{ is absolutely continuous.} \end{aligned}$$

Therefore f is differentiable at all points in (a, b), and $f'(x) = g(x)$ for $x \in (a, b)$. a similar proof works at the endpoints $x = a$ and $x = b$ if we take appropriate limits from the left or right.

Second proof. Since g is continuous, its antiderivative $F(x) = \int_a^x g(t)\, dt$ is absolutely continuous, differentiable at all points, and satisfies $F'(x) = g(x)$ for every $x \in [a, b]$. Hence $(F - f)' = F' - f' = g - g = 0$ a.e. An absolutely continuous function whose derivative is zero almost everywhere must be a constant. Therefore $F - f$ is constant, so $f = F + c$ and f is differentiable at all points. Also $f'(x) = F'(x) = g(x)$ for all $x \in [a, b]$.

7. Let Y be a dense subspace of a normed linear space X, and let Z be a Banach space. Let $L\colon Y \to Z$ be a bounded linear operator.

(a) Prove that there exists a unique bounded linear operator $\widetilde{L}\colon X \to Z$ whose restriction to Y is L. Prove that $\|\widetilde{L}\| = \|L\|$.

(b) Prove that if $L\colon Y \to \mathrm{range}(L)$ is a topological isomorphism (L is a bijection and L, L^{-1} are both continuous) then $\widetilde{L}\colon X \to \overline{\mathrm{range}(L)}$ is also a topological isomorphism.

<u>Solution</u>

(a) Fix any $f \in X$. Since Y is dense in X, there exist $g_n \in Y$ such that $g_n \to f$. Since L is bounded, we have $\|Lg_m - Lg_n\| \le \|L\|\,\|g_m - g_n\|$. But $\{g_n\}_{n\in\mathbb{N}}$ is Cauchy in X, so this implies that $\{Lg_n\}_{n\in\mathbb{N}}$ is Cauchy in Z. Since Z is a Banach space, we conclude that there exists an $h \in Z$ such that $Lg_n \to h$. Define $\widetilde{L}f = h$.

To see that $\widetilde{L}$ is well-defined, suppose that we also had $g_n' \to f$ for some $g_n' \in Y$. Then $\|Lg_n' - Lg_n\| \le \|L\|\,\|g_n' - g_n\| \to 0$. Since $Lg_n \to h$, it follows that $Lg_n' = Lg_n + (Lg_n' - Lg_n) \to h + 0 = h$. Thus $\widetilde{L}$ is well-defined, and similarly it is linear.

To see that $\widetilde{L}$ is an extension of L, suppose that $g \in Y$ is fixed. If we set $g_n = g$, then $g_n \to g$ and $Lg_n \to Lg$, so by definition we have $\widetilde{L}g = Lg$. Hence the restriction of $\widetilde{L}$ to Y is L. Consequently,

$$\|\widetilde{L}\| = \sup_{f\in X,\,\|f\|=1} \|\widetilde{L}f\| \ge \sup_{f\in Y,\,\|f\|=1} \|\widetilde{L}f\| = \sup_{f\in Y,\,\|f\|=1} \|Lf\| = \|L\|.$$

Now suppose that $f \in X$. Then there exist $g_n \in Y$ such that $g_n \to f$ and $Lg_n \to \widetilde{L}f$, so

$$\|\widetilde{L}f\| = \lim_{n\to\infty} \|Lg_n\| \le \lim_{n\to\infty} \|L\|\,\|g_n\| = \|L\|\,\|f\|.$$

Hence $\|\widetilde{L}\| \le \|L\|$. Combining this with the opposite inequality derived above, we conclude that $\|\widetilde{L}\| = \|L\|$.

Finally, we must show that $\widetilde{L}$ is unique. Suppose that $A \in \mathcal{B}(X, Y)$ also satisfied $A|_Y = L$. Then $Af = \widetilde{L}f$ for all $f \in Y$. Since Y is dense, this extends by continuity to all $f \in X$, which implies that $A = \widetilde{L}$.

(b) Suppose that $L\colon Y \to \mathrm{range}(Y)$ is a topological isomorphism. We already know that $\widetilde{L}\colon X \to \mathrm{range}(\widetilde{L})$ is bounded. We need to show that $\widetilde{L}$ is injective, that $\widetilde{L}^{-1}\colon \mathrm{range}(\widetilde{L}) \to X$ is bounded, and that $\mathrm{range}(\widetilde{L}) = \overline{\mathrm{range}(L)}$.

Fix any $f \in X$. Then there exist $g_n \in Y$ such that $g_n \to f$ and $Lg_n \to \widetilde{L}f$. Since L is a topological isomorphism, $\|g_n\| = \|L^{-1}Lg_n\| \le \|L^{-1}\|\,\|Lg_n\|$. Hence

$$\|\widetilde{L}f\| = \lim_{n\to\infty} \|Lg_n\| \ge \lim_{n\to\infty} \frac{\|g_n\|}{\|L^{-1}\|} = \frac{\|f\|}{\|L^{-1}\|}.$$

Consequently, $\widetilde{L}$ is injective and for any $h \in \mathrm{range}(\widetilde{L})$ we have

$$\|\widetilde{L}^{-1}h\| \le \|L^{-1}\|\,\|\widetilde{L}(\widetilde{L}^{-1}h)\| = \|L^{-1}\|\,\|h\|.$$

Therefore $\widetilde{L}^{-1}\colon \mathrm{range}(\widetilde{L}) \to X$ is bounded.

It remains only to show that the range of $\widetilde{L}$ is the closure of the range of L. If $f \in X$, then by definition there exist $g_n \in Y$ such that $g_n \to f$ and $Lg_n \to \widetilde{L}f$. Hence $\widetilde{L}f \in \overline{\text{range}(L)}$, so $\text{range}(\widetilde{L}) \subseteq \overline{\text{range}(L)}$.

On the other hand, suppose that $h \in \overline{\text{range}(L)}$. Then there exist $g_n \in Y$ such that $Lg_n \to h$. Since $\widetilde{L}^{-1}$ is bounded and $\widetilde{L}$ extends L, we conclude that $g_n = \widetilde{L}^{-1}(Lg_n) \to \widetilde{L}^{-1}(h)$. Hence $f = \widetilde{L}^{-1}(h)$, so $f \in \text{range}(\widetilde{L})$.

8. Assume that $E \subset \mathbb{R}^d$ is Lebesgue measurable and $m(E) > 0$, where m denotes Lebesgue measure. Show that there exists a point $x \in E$ such that for every $\delta > 0$ we have $m(E \cap B_\delta(x)) > 0$. Here, $B_\delta(x)$ denotes the open ball with center x and radius δ.

<u>Solution</u>

Suppose on the contrary that for every $x \in E$ there exists $\delta_x > 0$ such that $m(E \cap B_{\delta_x}(x)) = 0$. Set

$$\mathcal{F} := \{\bar{B}_\delta(x) \mid x \in E, 0 < \delta < \min\{\delta_x, 1\}\}.$$

By Vitali's Covering Lemma there exists $\mathcal{G}$, a countable family of disjoint balls in $\mathcal{F}$, such that

$$\cup_{B \in \mathcal{F}} B \subset \cup_{B \in \mathcal{G}} \hat{B}$$

where $\hat{\bar{B}}_\delta(x) = \bar{B}_{5\delta}(x)$. As $E \subset \cup_{B \in \mathcal{F}} B$ we have $E \subset \cup_{B \in \mathcal{G}} \hat{B}$ and so,

$$m(E) \leq \sum_{B \in \mathcal{G}} m(\hat{B}) = 0.$$

This contradicts the fact that $m(E) > 0$.