ALGEBRA COMPREHENSIVE EXAM QUESTIONS
FALL 2014

Choose any five problems. If you attempt more than five, indicate clearly which five are to be graded.

(1) Show that a group of order 80 cannot be simple.

Solution: Let G be a group of order 80. We have $80 = 16 \cdot 5 = 2^4 \cdot 5$. Let n_5 be the number of Sylow 5-subgroups. Then $n_5 \equiv 1 \mod 5$ and n_5 divides 16, so either $n_5 = 1$ or $n_5 = 16$. In the first case, the Sylow 5-subgroup is normal. In the second case, since a Sylow 5-subgroup is generated by any nontrivial element, any distinct Sylow 5-subgroups H, H' have $H \cap H' = \{1\}$. Therefore there are $16 \cdot 4 = 64$ elements of G of order 5 (and 1 of order 1), so there are at most $15 = 80 - 65$ elements of G of even order. It follows that there is a unique Sylow 2-group (of order 16).

(2) Let R be a commutative ring with 1. Show that the set of all nilpotent elements is an ideal of R.

Solution: Let I be the set of all nilpotent elements in R. Let $x, y \in I$, and choose $n \gg 0$ such that $x^n = y^n = 0$. Let $N = 2n - 1$. We have

$$(x-y)^N = \sum_{i=0}^{N} (-1)^{N-i} \binom{N}{i} x^i y^{N-i} = 0$$

because for every $0 \le i \le N$ either $i \ge n$ or $i \le n-1$ in which case $N - i \ge N - (n-1) = n$. Hence I is an additive subgroup of R. If $x \in I$ and $y \in R$ then $(yx)^n = y^n x^n = 0$ for $n \gg 0$, so I is an ideal.

(3) Let R and S be commutative rings with 1 and let $f : R \to S$ be a ring homomorphism. Prove that if R is a field then either f is injective or $S = 0$.

Solution: Suppose that f is not injective. Then $\ker(f)$ contains a nonzero element $x \in R$. Since $\ker(f)$ is an ideal in R and R is a field, we have $1 = x^{-1}x \in \ker(f)$, so $f(1) = 0$. A ring homomorphism by definition takes 1 to the multiplicative identity, so $0 = 1$ in S, and hence $y = 1 \cdot y = 0 \cdot y = 0$ for all $y \in S$.

(4) Let V be a finite-dimensional complex vector space. A linear operator $T : V \to V$ is called *nilpotent* if $T^m = 0$ for some $m \in \mathbf{N}$ (i.e. $T^m v = 0$ for all $v \in V$). Show that if T is nilpotent, then $T^n = 0$, where n is the dimension of V.

Solution: First we claim that all eigenvalues of T are zero. If λ is an eigenvalue with eigenvector v then

$$0 = T^m v = \lambda T^{m-1} v = \lambda^2 T^{m-2} v = \cdots = \lambda^m v;$$

hence $\lambda^m = 0$, so $\lambda = 0$, which establishes the claim. Let $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of T. The roots of f are the eigenvalues of T, so f has no nonzero roots, and hence $f(t) = t^n$. By the Cayley–Hamilton theorem, $f(T) = T^n = 0$.

(5) Let $f(X) = (X^7 - 1)/(X - 1) = X^6 + X^5 + \cdots + 1$. Prove that f is irreducible over $\mathbf{F}_3$ but not over $\mathbf{F}_7$.

Solution: Let $\alpha \in \bar{\mathbf{F}}_3$ be a primitive 7th root of unity and let g be the irreducible factor of f such that $g(\alpha) = 0$. Consider the field extension $\mathbf{F}_3[\alpha]$. Let $\varphi : \mathbf{F}_3[\alpha] \to \mathbf{F}_3[\alpha]$, $x \mapsto x^3$ be the Frobenius homomorphism. Since $g(x)^3 = g(x^3)$ we have $g(\varphi(\alpha)) = g(\alpha)^3 = g(\alpha) = 0$, so that $\varphi^i(\alpha)$ is a root of g for all i. Observe that

$$\alpha,\ \varphi(\alpha) = \alpha^3,\ \varphi^2(\alpha) = \alpha^9 = \alpha^2,\ \varphi^3(\alpha) = \alpha^6,\ \varphi^4(\alpha) = \alpha^{18} = \alpha^4,\ \varphi^5(\alpha) = \alpha^{12} = \alpha^5$$

are all distinct elements of $\mathbf{F}_3[\alpha]$. Therefore g has 6 distinct roots in $\mathbf{F}_3[\alpha]$ and thus $g = f$ as desired.

Over $\mathbf{F}_7$ we note that $x^7 - 1 = (x-1)^7$.

(6) Let G be a group of order 140 and H be a subgroup of of G of index 4. Show that H is normal in G.

Solution: Let G act on the set of (left) cosets of H by (left) multiplication. This action leads to a homomorphism $\varphi : G \to S_4$, the symmetric group on 4 elements. Since the stabilizer of H is H it follows that the kernel of φ is a subgroup of H. On the other hand, since 5 and 7 don't divide the order of $S_4 = 24$, we have $|\ker(\varphi)| \geq 5 \cdot 7 = 35 = |H|$. Therefore we have $H = \ker\varphi$, so H is normal in G.

(7) Let M be a finitely generated free abelian group of rank n and let $N \subset M$ be a subgroup such that M/N is finite. Prove that there exists a basis $x_1, \dots, x_n$ of M and positive integers $a_1, \dots, a_n$ such that $a_1x_1, \dots, a_nx_n$ is a basis for N.

Solution: If $r = [M : N]$ then $\mathbf{Z}^n \cong rM \subset N \subset M \cong \mathbf{Z}^n$, so $n \leq \operatorname{rank}(N) \leq n$, i.e. $\operatorname{rank}(N) = n$. Let $x_1, \dots, x_n$ be a basis of M and let $y_1, \dots, y_n$ be a basis of N. For $i = 1, \dots, n$ we can write $y_i = c_{i1}x_1 + \cdots + c_{in}x_n$ with $c_{ij} \in \mathbf{Z}$. Let C be the matrix with ij-entry given by c_{ij}. The matrix C can be reduced to Smith Normal Form, which amounts to a sequence of change of basis operations on $\{y_i\}$ and $\{x_i\}$. The matrix in Smith Normal Form is diagonal and thus gives a basis of M whose multiples are a basis of N as desired.

(8) Let $F = \mathbf{Q}(\sqrt{2}, \sqrt{3})$ and define $T : F \to F$ by $T(x) = x \cdot \sqrt{2}$. Prove that T is a linear transformation of $\mathbf{Q}$-vector spaces, and find its characteristic polynomial.

Solution: Associativity and distributivity in F imply that $T(x+y) = Tx + Ty$ and $T(cx) = cTx$ for $x, y \in F$ and $c \in \mathbf{Q}$. A basis for F over $\mathbf{Q}$ is given by the elements $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$. With respect to this basis, the matrix for T is

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence $\det(tI - T) = (t^2 - 2)^2$.