Discrete Math Comprehensive Exam
Aug 26, 2015

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Please note that a complete solution of a problem is preferable to partial progress on several problems. Write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Let $n \geq 1$ be an integer. A ship carries $5n$ flags, n each of the colors red, green, white, blue and black. In order to communicate a signal to other ships these flags are placed on a vertical pole. Thus one flag will be on top, the second flag will be immediately below the first, the third flag will be immediately below the second, and so on. How many such signals consisting of a total of n flags have an even number of blue flags and an odd number of black flags?

2. Let T be a tree, and let $f : E(T) \to V(T)$ be a mapping such that $f(e)$ is incident with e for every $e \in E(T)$. Prove that there exists a vertex $v \in V(T)$ such that $f(e) = v$ for every edge $e \in E(T)$ incident with v.

3. Let G be a bipartite multigraph, and let Δ be its maximum degree. Prove that G has a matching saturating every vertex of degree Δ.

4. Let $g_1, g_2, \ldots, g_k$ be bounded real functions, let f be another real function, and let δ and ϵ be positive constants. Assume that $\max_{1\leq i\leq k}[g_i(x)-g_i(y)] > \delta$ whenever $f(x)-f(y) > \epsilon$. Prove that f is bounded.

5. The *crossing number* of a graph G is the minimum number of crossings among all drawings of G in the plane, so a planar graph has crossing number 0. Show that if G is a graph with n vertices and e edges with $e > 4n$, then the crossing number of G is at least $e^3/64n^2$.

6. Show that for every pair (g, r) of integers with $g, r \geq 3$, there is a graph G whose girth is larger than g and whose chromatic number is larger than r.

7. A hypergraph is *simple* if any two edges intersect in at most one point, and it is *t-uniform* if all edges have size t. Use the Lovász Local Lemma (symmetric form) to show that any 4-uniform simple hypergraph has chromatic number at most 10 if every vertex belongs to at most 63 edges.