Comprehensive Examination, February 2003

REAL ANALYSIS

Instructions: (1) Please do any 5 of the 7 problems. (2) Be sure to justify your assertions. Please provide careful and complete answers; partial progress towards many questions counts less than a complete answer to fewer questions. If you attempt more than five questions, specify which ones you want to be graded, otherwise the first five answered will be graded.

1. Proof or counterexample: If f is a nonnegative function in $L^1[0,1]$ and $\int_0^1 f(x)dx = 1$, then there exists a measurable set $A \subset [0,1]$ such that

$$\mu(A) = \frac{1}{2} \qquad \text{and} \qquad \int_A f = \frac{1}{2}$$

(μ denotes Lebesgue measure).

2. Let μ denote Lebesgue measure on $[0,1]$, ν counting measure on $[0,1]$, and $\mathcal{H}$ Hausdorff measure on $[0,1]$. Hausdorff measure is given by

$$\mathcal{H}(A) = \lim_{r \searrow 0} \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j, \operatorname{diam}(A_j) \leq r \right\}$$

on

$$\mathcal{M} = \{A \subset [0,1] : \mathcal{H}(A \cap B) + \mathcal{H}(B \backslash A) = \mathcal{H}(B) \text{ for every } B \subset [0,1]\}.$$

Here, $\operatorname{diam}(A_j) = \sup\{|x-y| : x, y \in A_j\}$ is the diameter of A_j.

(a) Give precise definitions for μ and ν.

(b) Determine which of the following assertions concerning absolute continuity are true and which are false on the intersection of the domains of the given measures:

$$\mu \ll \nu, \quad \mu \ll \mathcal{H}, \quad \nu \ll \mu, \quad \nu \ll \mathcal{H}, \quad \mathcal{H} \ll \mu, \quad \mathcal{H} \ll \nu.$$

(Justify your answers.)

(c) Is there a function f such that $\mu(A) = \int_A f \, d\nu$ for every $A \in \mathcal{M}$? (Justify your answer.)

3. Suppose $f_n, f \in L^1(-\infty,\infty)$, $\|f_n\|_{L^1(-\infty,\infty)} \le 1$ for all n, and $\int_E f_n \to \int_E f$ for every measurable set E. Prove that if g is a measurable function with $0 \le g \le 1$ a.e., then $\int f_n g \to \int fg$.

4.(a) Give a counterexample to the statement: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable functions on $(-\infty,\infty)$ satisfying $f_1 > f_2 > \cdots > 0$ a.e. and $f_n \to f$ a.e., then

$$\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f.$$

(b) Add one additional hypothesis to the statement of part (a) (do not otherwise modify the given hypotheses) and prove the resulting assertion.

5. Denote the class of absolutely summable sequences of real numbers by l^1. Let $\alpha = \{\alpha_n\}_{n=1}^{\infty}$, $\beta = \{\beta_n\}_{n=1}^{\infty} \in l^1$ with $\alpha_n, \beta_n > 0$ for all n. Assume $\|\beta\| = 1$. Show that

$$\prod_{n=1}^{\infty} \alpha_n^{\beta_n} \le \sum_{n=1}^{\infty} \alpha_n \beta_n < \infty.$$

6. State carefully the Riesz Representation Theorem for linear functionals on L^p, $1 \le p < \infty$, and prove the uniqueness of Riesz representation.

7. Let $A = \big\{ f \in L^1[0,1] : |f(x)| \ge 1 \text{ a.e.} \big\}$ True or False

(a) A is norm closed in $L^1[0,1]$.

(b) A is weakly closed in $L^1[0,1]$.

Comprehensive Examination, Spring 2003

ALGEBRA

Instructions: (1) Please do any 5 of the 7 problems. (2) Be sure to justify your assertions. Please provide careful and complete answers; partial progress towards many questions counts less than a complete answer to fewer questions. If you attempt more than five questions, specify which ones you want to be graded, otherwise the first five answered will be graded.

1. Let G be a finite group. For any $x \in G$, let $Z(x) = \{g \in G : gx = xg\}$. Let $\mathcal{C}(G) = \{Z(x) : x \in G\}$. Prove the following statements.
   (a) If $|\mathcal{C}(G)| = 1$ then G is Abelian.
   (b) $|\mathcal{C}(G)| \neq 2$.
   (c) $|\mathcal{C}(G)| \neq 3$.

2. (a) Show that if H and K are normal subgroups of a group and $H \cap K = \{1\}$ where 1 is the identity, then $xy = yx$ for all $x \in H$ and $y \in K$.
   (b) Let G be a group of order pq, where $p < q$ and both p and q are prime numbers. Let P be a subgroup of G of order p and Q a subgroup of G of order q. Prove that Q is a normal subgroup of G, and if P is a normal subgroup of G then G is cyclic.

3. Let R be an integral domain, and let $R\{x\}$ denote the set of formal power series in x with coefficients in R. Then $R\{x\}$ is a commutative ring under the following operations:
$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n, \qquad \text{and}$$
$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n.$$
   Prove the following statements.
   (a) $I = (x)$, the principle ideal generated by x, is a prime ideal in $R\{x\}$.
   (b) I is a maximal ideal if and only if R is a field.

4. Let F be a finite field. Prove (from first principles) that there exist a prime number p and a positive integer n such that $|F| = p^n$.

5. Let $\mathbb{Z}_p[x]$ denote the polynomial ring with coefficients in $\mathbb{Z}_p$ (where p is a prime number) and let $f(x)$ be an irreducible polynomial over $\mathbb{Z}_p$ of degree $n > 0$. Show (from first principles) that $\mathbb{Z}_p[x]/(f(x))$ is a field with p^n elements. Here, $f(x))$ is the ideal in $\mathbb{Z}_p[x]$ generated by $f(x)$.

6. Prove that $\langle A, B\rangle = \mathrm{trace}(AB^T)$ defines an inner product on the space of $n \times n$ real matrices, and find the orthogonal complement of the subspace of all skew symmetric matrices.

7. Prove that diagonalizable matrices A and B can be simultaneously diagonalized (there exists a matrix S with $S^{-1}AS$ and $S^{-1}BS$ both diagonal) if and only if $AB = BA$.

SOLUTIONS: Algebra

1. **Proof of (a).** $\forall\ x, y \in G$ with $x \neq y$, we have $Z(x) = Z(y) = Z(1) = G$. Hence $x \in Z(y)$. Therefore $xy = yx$. So G is Abelian.

   Proof of (b). Suppose for a contradiction that $|\mathcal{C}(G)| = 2$. Let $x \in G$ such that $Z(x) \neq Z(1) = G$. Since $Z(x) \neq G$, there exists some $y \in G \backslash Z(x)$. That is, $xy \neq yx$. Hence, $x \notin Z(y)$. But this implies that $Z(y) \neq Z(x)$ and $Z(y) \neq G$, contradicting the assumption that $|\mathcal{C}(G)| = 2$.

   Proof of (c). Suppose for a contradiction that $|\mathcal{C}(G)| = 3$. Let $x \in G$ such that $Z(x) \neq Z(1) = G$. Since $Z(x) \neq G$, there exists some $y \in G \backslash Z(x)$. As in the proof of (b), we know that $Z(y) \neq Z(x)$ and $Z(y) \neq Z(1)$.

   Now consider the element xy. We will derive a contradiction by showing that $Z(xy) \notin \{Z(x), Z(y), Z(1)\}$. Since $Z(x) \neq Z(y)$, $xy \neq yx$. Therefore, $x(xy) \neq x(yx) = (xy)x$, and so, $Z(xy) \neq Z(x)$ and $Z(xy) \neq Z(1)$. Similarly, $(xy)y \neq (yx)y = y(xy)$ and so, $Z(xy) \neq Z(y)$. Thus $|\mathcal{C}(G)| \geq 4$, a contradiction.

2. **Proof of (a).** We consider $(yx)^{-1}xy = x^{-1}y^{-1}xy$. Since $x^{-1}y^{-1}x \in x^{-1}Kx = K$ (because K is normal), $x^{-1}y^{-1}xy \in K$. Since $y^{-1}xy \in y^{-1}Hy = H$ (because H is normal), $x^{-1}y^{-1}xy \in H$. Since $H \cap K = \{1\}$, we have $x^{-1}y^{-1}xy = 1$. Hence, $xy = yx$ for all $x \in H$ and $y \in K$.

   Proof of (b). First, we show that Q is a normal subgroup of G. Since Q is of order q which is a prime number and q^2 does not divide $pq = |G|$, Q is a Sylow subgroup of G. Hence by Sylow's theorem, $|\{g^{-1}Qg : g \in G\}| \equiv 1 (\text{mod } q)$. We claim that $|\{g^{-1}Qg : g \in G\}| = 1$. For otherwise, $|\{g^{-1}Qg : g \in G\}| \geq q + 1$. Since $\forall\ g, h \in G$, either $g^{-1}Qg = h^{-1}Qh$ or $g^{-1}Qg \cap h^{-1}Qh = \{1\}$, we see that $|G| \geq (q+1)(q-1) + 1 = q^2 > pq = |G|$, a contradiction. So $|\{g^{-1}Qg : g \in G\}| = 1$, and hence, $g^{-1}Qg = Q\ \forall\ g \in G$. Therefore, Q is a normal subgroup of G.

   Now assume that P is normal in G. Since $p < q$ and both p and q are primes, $P \cap Q = \{1\}$. Therefore, by (a), $\forall\ x \in P$ and $y \in Q$, $xy = yx$. Let $P = \langle x \rangle$ and $Q = \langle y \rangle$. Consider the element xy in G. Clearly $(xy)^{pq} = x^{pq}y^{pq} = 1$. Also $(xy)^p = x^p y^p = y^p \neq 1$ and $(xy)^q = x^q y^q = x^q \neq 1$ (again because p, q are primes and $p < q$). So the order of xy is pq, and hence $G = \langle xy \rangle$ (because $|G| = pq$).

3. **Proof of (a).** Suppose $\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) \in I$. Then $a_0 b_0 = 0$. Therefore, since R is an integral domain, $a_0 = 0$ or $b_0 = 0$. If $a_0 = 0$ then $\sum_{n=0}^{\infty} a_n x^n \in I$, and if $b_0 = 0$ then $\sum_{n=0}^{\infty} b_n x^n \in I$. So I is a prime ideal.

   Proof of (b). Suppose I is a maximal ideal. Let $r \in R$ and assume $r \neq 0$. Then the ideal generated by x and r must be equal to $R\{x\}$. Therefore, there exist $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ in $R\{x\}$ such that $x \cdot \sum_{n=0}^{\infty} a_n x^n + r \cdot \sum_{n=0}^{\infty} b_n x^n = 1$. This implies that $r b_0 = 1$. Hence, r has an inverse. Since r is arbitrary, R is a field.

   Now assume that R is a field. Let $\sum_{n=0}^{\infty} a_n x^n \in R\{x\} - I$. Then $a_0 \neq 0$. Consider the ideal I' generated by x and $\sum_{n=0}^{\infty} a_n x^n$, which contains the element $\sum_{n=1}^{\infty} (-a_n) x^n + \sum_{n=0}^{\infty} a_n x^n = a_0$. Since $a_0 \neq 0$ and since R is a field, $1 \in I'$. So $I' = R\{x\}$. Since $\sum_{n=0}^{\infty} a_n x^n$ is arbitrary, I must be a maximal ideal in $R\{x\}$.

4. **Proof.** Let 0 and 1 denote the additive identity and multiplicative identity of F, respectively. Since F is finite, there is a positive integer p such that $\underbrace{1 + \cdots + 1}_{p} = 0$ (because the sequence $1, 1{+}1, 1{+}1{+}1, \ldots$ must repeat). Choose such p to be minimum.

   Then p must be a prime number. For otherwise, there exist integers m, q such that $2 \leq m, q < p$ and $p = mq$. Then $\underbrace{(1 + \cdots + 1)}_{m} \underbrace{(1 + \cdots + 1)}_{q} = \underbrace{1 + \cdots + 1}_{p} = 0$. Since F is a field, either $\underbrace{1 + \cdots + 1}_{m} = 0$ or $\underbrace{1 + \cdots + 1}_{q} = 0$, contradicting the choice of p.

   Let $\mathbb{Z}_p = \left\{0, 1, 1+1, \ldots, \underbrace{1 + \cdots + 1}_{p-1}\right\}$. Then $\mathbb{Z}_p$ is closed under both operations of F. So $\mathbb{Z}_p$ is a subfield of F.

   We may view F as an extension of $\mathbb{Z}_p$. Since F is finite, $[F : \mathbb{Z}_p]$ is finite. Let $n = [F : \mathbb{Z}_p]$, and let $v_1, \ldots, v_n$ be a basis of F as a vector space over $\mathbb{Z}_p$. Then

$F = \{c_1 v_1 + \cdots + c_n v_n : c_i \in \mathbb{Z}_p, i = 1, \ldots, n\}$. Hence $|F| = p^n$.

5. **Proof.** First, we show that $\mathbb{Z}_p[x]/(f(x))$ under addition is an Abelian group. Let $[g(x)]$, $[h(x)]$, $[k(x)] \in \mathbb{Z}_p[x]/(f(x))$. Then

$$
\begin{aligned}
&(1) \quad ([g(x)] + [h(x)]) + [k(x)] = [g(x) + h(x)] + [k(x)] \\
&\qquad = [(g(x) + h(x)) + k(x)] = [g(x) + (h(x) + k(x))] \\
&\qquad = [g(x)] + [h(x) + k(x)] = [g(x)] + ([h(x)] + [k(x)]), \\
&(2) \quad [g(x)] + [h(x)] = [g(x) + h(x)] = [h(x) + g(x)] = [h(x)] + [g(x)], \\
&(3) \quad [0] + [g(x)] = [0 + g(x)] = [g(x)], \quad \text{and} \\
&(4) \quad [-g(x)] + [g(x)] = [-g(x) + g(x)] = [0],
\end{aligned}
$$

where

$$
-g(x) = \sum_{i=0}^{n} (-a_i) x^i \qquad \text{if } g(x) = \sum_{i=0}^{n} a_i x^i.
$$

Next, we prove distribution property.

$$
\begin{aligned}
[g(x)]([h(x)] + [k(x)]) &= [g(x)][h(x) + k(x)] \\
&= [g(x)(h(x) + k(x))] \\
&= [g(x)h(x) + g(x)k(x)] \\
&= [g(x)h(x)] + [g(x)k(x)] \\
&= [g(x)][h(x)] + [g(x)][k(x)].
\end{aligned}
$$

Now we show that $\mathbb{Z}_p[x]/(f(x)) - \{[0]\}$ under multiplication is an Abelian group. Note (1), (2), (3) are the same as for additions (with [1] replacing [0] in (3)). We only show (4), the existence of a multiplicative inverse.

For each $[g(x)] \in \mathbb{Z}_p[x]/(f(x))$, we may assume the degree of $g(x)$ is less than n. For, if degree of $g(x)$ is $\geq n$, then by division algorithm, there exists $q(x)$, $r(x) \in \mathbb{Z}_p[x]$ such that $g(x) = q(x)f(x) + r(x)$ and degree of $r(x)$ is less than n. Hence $[g(x)] = [r(x)]$ (since $[f(x)] = [0]$) and we could use $r(x)$ instead of $g(x)$.

Let $[g(x)] \in \mathbb{Z}_p[x]/(f(x)) - \{[0]\}$. Since $f(x)$ is irreducible and because $g(x) \neq 0$ and the degree of $g(x)$ is less than n, we have $\gcd(g(x), f(x)) = 1$. By Euclidean algorithm, there exist $s(x)$, $t(x) \in \mathbb{Z}_p[x]$ such that $s(x)g(x) + t(x)f(x) = 1$. Therefore $[s(x)][g(x)] + [t(x)][f(x)] = [1]$. Since $[f(x)] = [0]$, $[s(x)][g(x)] = [1]$. So $[g(x)]$ has a multiplicative inverse.

Finally, we show that $\mathbb{Z}_p[x]/(f(x))$ has p^n elements. For any $[g(x)]$, $[h(x)] \in \mathbb{Z}_p[x]/(f(x))$, we see that $[g(x)] = [h(x)]$ if and only if $g(x) = h(x)$ (from previous assumptions that both $g(x)$ and $h(x)$ have degree $< n$). Therefore, since coefficients of $g(x)$ are in $\mathbb{Z}_p$, we see that $\mathbb{Z}_p[x]/(f(x))$ has p^n elements.

6. **Proof of (a).** It is easy to see that

$$\langle A, B \rangle = \text{trace}(AB^T)$$

is symmetric and linear in the first variable. Further, AA^T is positive semidefinite, hence diagonalizable with real nonnegative eigenvalues. Since trace(AA^T) is the sum of the eigenvalues we have $\langle A, A \rangle \geq 0$ for all A, and $\langle A, A \rangle = 0$ if and only if all eigenvalues are 0, in which case $AA^T = 0$. [AA^T is diagonalizable]. Since rank$(AA^T) =$ rank A, A is 0 as well.

Proof of (b). We claim the orthogonal complement of the skew-symmetric matrices, SK, is the subspace, S, of symmetric matrices. If $A = (a_{ij})$ is symmetric and $B = (b_{ij})$ is skew symmetric, then

$$\begin{aligned} \langle A, B \rangle &= \text{trace}(AB^T) \\ &= \sum_i \cdot \sum_j a_{ij} b_{ij} \\ &= \sum_{i,j} a_{ij} b_{ij} = 0 \end{aligned}$$

since $a_{ij} = a_{ji}$ and $b_{ij} = -b_{ji}$ for all $i \neq j$ and $b_{ii} = 0$ for all i. Thus $S \subset SK^\perp$. Now $\dim(SK) = \dfrac{(n-1)n}{2}$ so $\dim(SK^\perp) = n^2 - \dfrac{(n-1)n}{2} = \dfrac{n^2}{2} - \dfrac{n}{2}$. But $\dim(S) = \dfrac{(n-1)(n)}{2} + n = \dfrac{n^2}{2} - \dfrac{n}{2}$ and since $\dim(S) = \dim(SK^\perp)$, we have $S = SK^\perp$.

7. **Proof.** If $D_1 = S^{-1}AS$ and $D_2 = S^{-1}BS$ are diagonal, then $AB = SD_1S^{-1}SD_2S^{-1} = SD_1D_2S^{-1} = SD_2D_1S^{-1} = BA$. The other direction is harder. Suppose $AB = BA$. We will use the following fact.

A matrix is diagonalizable if and only if its minimal polynomial factors as $(x - \lambda_1) \cdots (x - \lambda_k)$ where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues.

We can assume (by diagonalizing A) that

$$A = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & 0 & & \\ & & & \ddots & & & \\ & 0 & & & \lambda_k & & \\ & & & & & \ddots & \\ & & & & & & \lambda_k \end{pmatrix}.$$

where each diagonal block $\begin{pmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{pmatrix}$ corresponds to an invariant subspace $E_j = \{\text{eigenvectors of } A \text{ with eigenvalue } \lambda_j\}$ and

$$\mathbb{R}^n = E_1 \oplus \cdots \oplus E_k.$$

Claim: Each E_i is an invariant subspace for B.

Proof: If $v \in E_i$, then $Av = \lambda_i v$. Therefore, $A(Bv) = BAv = B(\lambda_i v) = \lambda_i(Bv)$. Thus, $Bv \in E_i$. Therefore, B has block diagonal form

$$\begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{pmatrix}$$

(with each block B_j the same size as the corresponding λ_j block of A).

Claim: Each B_i is diagonalizable on E_i.

Proof: Note that the minimal polynomial m_B of B acts blockwise. That is,

$$M_B(B) = \begin{pmatrix} m_B(B_1) & & \\ & \ddots & \\ & & m_B(B_k) \end{pmatrix} = (0).$$

Therefore, if m_j is the minimal polynomial of B_j, then $m_j \mid m_B$. According to the fact stated at the beginning, m_j must consist of distinct factors $(x - \mu_1) \cdots (x - \mu_\ell)$. Therefore B_j is diagonalizable.

Pick a basis in each E_j that diagonalizes B_j. Combining these basis elements gives a basis for $\mathbb{R}^n$ that diagonalizes B. Since each basis element comes from one of the E_i, it is an eigenvector of A and A keeps the same (diagonal) form.

SOLUTIONS: Real Analysis

1. If $\int_0^{1/2} f(x)dx = 1/2$ we take $A = \left[0, \frac{1}{2}\right]$ and we're done. Otherwise, assume $\int_0^{1/2} f(x)dx < 1/2$. Then $\int_{1/2}^{1} f(x)dx > 1/2$, and the function

$$g(t) = \int_t^{t+1/2} f(x)dx \qquad \text{is continuous in } t,$$

so there exists t_0, $0 < t_0 < 1/2$ such that $g(t_0) = 1/2$. Taking $A = \left[t_0, t_0 + \frac{1}{2}\right]$, we are done. If $\int_0^{1/2} f(x)dx > 1/2$, the same argument applies.

2. **Proof of (a).**

$$\mu(A) = \inf\left\{\sum_{j=1}^{\infty}(b_j - a_j) : A \subset \bigcup_{j=1}^{\infty}[a_j, b_j]\right\}$$

on the same algebra $\mathcal{M}$ given for $\mathcal{H}$.

Counting measure is given by cardinality on all subsets.

Proof of (b). Claim: $\mu = \mathcal{H}$. If $\{I_j\}$ are intervals covering A and having $\sum|I_j| < \mu(A) + \epsilon$, then we can break them into subintervals of length less than r to see that $\mathcal{H}(A) \leq \mu(A) + \epsilon$. Since ϵ is arbitrary, we have half of the claim. To see the other half, simply note that the infimum in the definition of $\mathcal{H}$ is a nondecreasing function of r. Therefore, we can replace each A_j with an interval $I_j \supset A_j$ and having the same diameter. Therefore, $\mu(A) \leq \sum |I_j| \leq \mathcal{H}(A)$.

We see then that $\mathcal{H} = \mu \ll \nu$ because if $\nu(A) = 0$, then it doesn't have any points; $\mu \ll \mathcal{H}$ and $\mathcal{H} \ll \mu$ because they are equal; that $\nu \ll \mu$ or $\mathcal{H}$ is false because any countable set has Lebesgue measure zero but infinitely many points; take for example $\{1/n : n = 1, 2, \ldots\}$.

Proof of (c). If there were such a function, then

$$f(x) \equiv \int_{\{x\}} f\, d\nu = \mu\{x\} = 0.$$

Therefore $f \equiv 0$. But $1 = \mu[0,1] \neq \int_{[0,1]} f\, d\nu$. (There is no such function.)

The point here is that the Radon-Nikodym Theorem requires sigma finite measure spaces.

3. Let $\epsilon > 0$ and choose a simple function ϕ such that $\|\phi - g\|_\infty < \epsilon/3$. Then

$$\left| \int f_n \phi - \int f_n g \right| \leq \|f_n\|_1 \|\phi - g\|_\infty < \epsilon/3$$

and

$$\left| \int f \phi - \int f g \right| \leq \|f\|_1 \|\phi - g\| < \epsilon/3$$

[No problem: Take $\|f - g\|_\infty < \max\left(\frac{\epsilon}{3} \frac{\|\phi\|, \epsilon}{3\|f\|} \right)$.] Next, by linearity of $\int$, find N such that

$$n > N \Rightarrow \left| \int f_n \phi - \int f \phi \right| < \frac{\epsilon}{3}.$$

Then $n > N \Rightarrow$

$$\left| \int f_n g - \int f g \right| \leq \left| \int f_n g - \int f_n \phi \right| + \left| \int f_n \phi - \int f \phi \right| + \left| \int f \phi - \int f g \right| < 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

4. **Proof of (a).** Counterexample: Let $f \equiv 0$. Let

$$f_n = \begin{cases} 1/n & -\infty < x \leq n \\ \dfrac{1}{2} + \dfrac{1}{n} & n < x < \infty \end{cases}$$

Thus $f_n \searrow 0$, strictly decreasing,

$$\int f_n = \infty \quad \forall n, \qquad \text{yet } \int f = 0.$$

Proof of (b). Additional hypothesis $f_1 \in L^1(-\infty, \infty)$. This result follows immediately from LDCT.

5. Use Jensen's Inequality which relates $\phi\left(\int f\right)$ and $\int \phi \circ f$ where ϕ is a convex function and the integral is over a probability space (total measure 1). If you can't remember which way the inequality goes, just let f be constant on each half interval of the unit interval with Lebesgue measure, and the convexity of ϕ will tell you what must be true. In any event, you get

$$\phi\left(\int f\right) \leq \int \phi \circ f.$$

Notice that since $\|\beta\| = 1$, we can use the β_n to partition $[0,1]$ into disjoint subintervals I_n of length β_n. In this way, $\sum \beta_n \gamma_n = \int_{[0,1]} f$ where f is defined to have the (constant) value γ_n on the interval I_n of length β_n. This turns sums into integrals, which in one way or another, is probably the easiest thing to do in this problem.

Now, let's start by taking log of the left side:

$$\log \prod_{n=1}^{N} \alpha_n^{\beta_n} = \sum_{n=1}^{N} \beta_n \log \alpha_n$$
$$= \int_{\bigcup_{j=1}^{N} I_j} \log \left[\sum_{n=1}^{N} \alpha_n \chi_{I_n} \right].$$

Applying exp to both sides and applying Jensen, we get

$$\prod_{n=1}^{N} \alpha_n^{\beta_n} \leq \int_{\bigcup_{j=1}^{N} I_j} \sum_{n=1}^{N} \alpha_n \chi_{I_n}$$
$$\nearrow \sum_{n=1}^{\infty} \alpha_n \beta_n.$$

Taking $\sum_{n=1}^{N} \alpha_n \beta_n$ separately, we see by Hölder's inequality that this is always smaller than $\|\alpha\| \|\beta\| < \infty$. This shows that all quantities involved are finite.

6. **The Reisz Representation Theorem:** If Λ is a bounded linear functional on L^p for some $p \in [1, \infty)$, then there is a unique $g \in L^q$ (where $q = 1 - 1/p$ when $p \neq \infty$ and $p = 1$ otherwise) such that

$$\Lambda(f) = \int fg$$

for all $f \in L^p$.

To see the uniqueness, assume there is some $\tilde{g}$ with

$$\int fg = \int f\tilde{g}$$

for all $f \in L^p$. If the measure of $\{x : g(x) \neq \tilde{g}(x)\}$ is nonzero, then we may assume that for some $\epsilon, N > 0$ the set $A = \{x \in [-N, N] : g(x) - \tilde{g}(x) > \epsilon\}$ has positive measure. Note that $f = \chi_A \in L^p$, but $\displaystyle\int fg - \int f\tilde{g} > \mu(A)\epsilon > 0$. This is a contradiction.

7. **Proof of (a).** If $y \notin A$, then there is $\epsilon > 0$ and set E of positive measure such that

$$|g(x)| < 1 - \epsilon \qquad \forall\, x \in E.$$

Then for $f \in A$ we have

$$\|f - g\|_1 = \int_0^1 |f - g| \geq \int_E |f - g| \geq \epsilon\mu(E)$$

Thus $\|f - g\| < \epsilon\mu(E) \Rightarrow f \notin A \Rightarrow \tilde{A}$ is open $\Rightarrow A$ is norm chosed in L^1.

Proof of (b). A is not weakly closed in L^1. Consider $f_0(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 \leq s \leq 1 \end{cases}$, extended periodically and $f_n(x) = f(2^n x)$. Then $\forall\ g \in L^\infty$, $\displaystyle\int f_n g \to 0$, so $f_n \to 0$ weakly, but $f \equiv 0 \notin A$.