Analysis Comprehensive Exam Questions
Spring 2008

1. Consider a sequence of functions $f_n \in L^1(\mathbb{R}^d)$ for $n = 0, 1, \ldots$, with

$$C := \sup_{n \geqslant 0} \int_{\mathbb{R}^d} |f_n| \, dx < \infty.$$

Suppose that the following assumptions are satisfied:

(i) $f_n \to f_0$ in measure as $n \to \infty$;

(ii) for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|A| \leqslant \delta \quad \Longrightarrow \quad \sup_{n \geqslant 0} \int_A |f_n| \, dx \leqslant \varepsilon; \tag{1}$$

(iii) for all $\varepsilon > 0$ there exists a set $K \subseteq \mathbb{R}^d$ with $|K| < \infty$ such that

$$\sup_{n \geqslant 0} \int_{\mathbb{R}^d \setminus K} |f_n| \, dx \leqslant \varepsilon. \tag{2}$$

Prove that $f_n \to f_0$ strongly (i.e., in norm) in $L^1(\mathbb{R}^d)$. Also show that all three conditions are necessary, by constructing three counterexamples, each of which satisfies two of the hypotheses (i), (ii), (iii) but not the third, and for which f_n does not converge strongly to f_0.

Solution

Let $\varepsilon > 0$ be given. By assumption, we can find $\delta > 0$ and $K \subseteq \mathbb{R}^d$ such that (1) and (2) are satisfied. Choose $\alpha > 0$ large enough such that $C/\alpha \leqslant \delta$ and define

$$A_n := \Big\{ x \in \mathbb{R}^d \colon |f_n(x)| > \alpha \Big\} \quad \text{for all } n \geqslant 0.$$

By Chebyshev's inequality, we obtain

$$|A_n| \leqslant \frac{1}{\alpha} \int_{\mathbb{R}^d} |f_n| \, dx \leqslant \delta \quad \text{for all } n \geqslant 0,$$

which implies by hypothesis (ii) that

$$\int_{A_m} |f_n| \, dx \leqslant \varepsilon \quad \text{for all } m, n \geqslant 0. \tag{3}$$

We can now decompose

$$\begin{aligned} &\int_{\mathbb{R}^d} |f_n - f_0| \, dx \\ &= \int_{\mathbb{R}^d \setminus K} |f_n - f_0| \, dx + \int_{A_n \cup A_0} |f_n - f_0| \, dx + \int_{K \setminus (A_n \cup A_0)} |f_n - f_0| \, dx. \end{aligned} \tag{4}$$

The first term in (4) can be estimated as

$$\int_{\mathbb{R}^d \setminus K} |f_n - f_0| \, dx \leqslant \int_{\mathbb{R}^d \setminus K} \Big\{ |f_n| + |f_0| \Big\} \, dx \leqslant 2\varepsilon,$$

by choice of K. For the second term we use (3) and obtain

$$\int_{A_n \cup A_0} |f_n - f_0|\, dx \leqslant \int_{A_n \cup A_0} \Big\{ |f_n| + |f_0| \Big\}\, dx \leqslant 4\varepsilon.$$

Now notice that on $\mathbb{R}^d \setminus (A_n \cup A_0)$ we have $|f_n| \leqslant \alpha$ and $|f_0| \leqslant \alpha$. Remember the definition of convergence in measure: for every $\gamma > 0$ we have

$$\lim_{n \to \infty} \Big| \Big\{ x \in \mathbb{R}^d \colon |f_n(x) - f_0(x)| \geqslant \gamma \Big\} \Big| = 0.$$

Let $\gamma := \varepsilon / |K|$, which is positive because $|K|$ is finite. Then there exists an index $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have the estimate

$$\Big| \Big\{ x \in \mathbb{R}^d \colon |f_n(x) - f_0(x)| \geqslant \varepsilon/|K| \Big\} \Big| \leqslant \frac{\varepsilon}{\alpha}.$$

To simplify notation, let $K' := K \setminus (A_n \cup A_0)$. Then

$$\begin{aligned} &\int_{K'} |f_n - f_0|\, dx \\ &\quad = \int_{K' \cap \{x \colon |f_n - f_0| \geqslant \varepsilon/|K|\}} |f_n - f_0|\, dx + \int_{K' \setminus \{x \colon |f_n - f_0| < \varepsilon/|K|\}} |f_n - f_0|\, dx \\ &\quad \leqslant 2\alpha \frac{\varepsilon}{\alpha} + \frac{\varepsilon}{|K|} |K| = 3\varepsilon. \end{aligned}$$

Combining all estimates, we obtain

$$\int_{\mathbb{R}^d} |f_n - f_0|\, dx \leqslant 9\varepsilon \quad \text{for all } n \geqslant N.$$

Since $\varepsilon > 0$ was arbitrary, we have proved that $f_n \to f_0$ in $L^1(\mathbb{R}^d)$.

The necessity of condition (i) follows from

$$f_n(x) := \begin{cases} \sin(nx), & \text{if } x \in [0,1], \\ 0, & \text{otherwise,} \end{cases}$$

which is compactly supported and bounded, hence satisfies (ii) and (iii), but does not converge strongly in $L^1(\mathbb{R})$.

The necessity of condition (ii) follows from

$$f_n(x) := \begin{cases} n, & \text{if } x \in [0, 1/n], \\ 0, & \text{otherwise,} \end{cases}$$

which is compactly supported and converges in measure to the zero function, hence satisfies (i) and (iii), but does not converge strongly in $L^1(\mathbb{R})$.

The necessity of condition (iii) follows from

$$f_n(x) := \begin{cases} 1/n, & \text{if } x \in [0, n], \\ 0, & \text{otherwise,} \end{cases}$$

which is bounded and converges in measure to the zero function, hence satisfies (i) and (ii), but does not converge strongly in $L^1(\mathbb{R})$. □

2. Let $|\cdot|_e$ denote exterior Lebesgue measure on $\mathbb{R}$. Suppose that E is a subset of $\mathbb{R}$ with $|E|_e < \infty$. Show that E is Lebesgue measurable if and only if for every $\varepsilon > 0$ we can write $E = (S \cup N_1) \setminus N_2$, where S is a finite union of nonoverlapping intervals and $|N_1|_e$, $|N_2|_e < \varepsilon$.

Solution

$\Rightarrow$. Suppose that E is Lebesgue measurable, and fix $\varepsilon > 0$. Then there exists an open $G \supseteq E$ such that $|G \setminus E|_e < \varepsilon$. Since G is open, there exist nonoverlapping open intervals I_k such that $G = \cup_{k=1}^\infty I_k$. Since $\sum_{k=1}^\infty |I_k| = |G| < \infty$, we can choose M large enough that $\sum_{k=M+1}^\infty |I_k| < \varepsilon$. Let

$$S = \bigcup_{k=1}^{M} I_k, \qquad N_1 = E \setminus S, \qquad N_2 = S \setminus E.$$

Note that S is a finite union of nonoverlapping intervals. Since $N_1 = E \setminus S \subseteq G \setminus S$, we have

$$|N_1|_e \le |G \setminus S| \le \left| \bigcup_{k=M+1}^{\infty} I_k \right| \le \sum_{k=M+1}^{\infty} |I_k| < \varepsilon.$$

Finally, $N_2 = S \setminus E \subseteq G \setminus E$, so

$$|N_2|_e \le |G \setminus E|_e < \varepsilon.$$

$\Leftarrow$. Assume that for any $\varepsilon > 0$ we can write $E = (S \cup N_1) \setminus N_2$, where S is a finite union of nonoverlapping intervals and $|N_1|_e$, $|N_2|_e < \varepsilon$. Since S is measurable, there exists an open set $U \supseteq S$ such that $|U \setminus S| < \varepsilon$. Although we don't know that N_1 is measurable, we can find an open set $V \supseteq N_1$ such that $|V| \le |N_1|_e + \varepsilon$. Consequently,

$$|V| \le |N_1|_e + \varepsilon < 2\varepsilon.$$

Let $G = U \cup V$. Then G is open, and since $U \supseteq S$ and $V \supseteq N_1$, we have that $G \supseteq S \cup N_1 \supseteq E$. After some tedious set-theoretic calculations, we see that

$$G \setminus E \;\subseteq\; (U \setminus S) \cup V \cup N_2.$$

Therefore

$$|G \setminus E|_e \le |U \setminus S| + |V| + |N_2|_e \le \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon.$$

Therefore E is measurable. $\square$

3. Prove directly the following special case of the Banach–Alaoglu theorem: if $\mathcal{X}$ is a separable, normed space and $\mathcal{X}^*$ is its dual space, then the set

$$B^* := \Big\{ f \in \mathcal{X}^* \colon \|f\|_* \leqslant 1 \Big\}$$

(with $\|\cdot\|_*$ the induced norm on $\mathcal{X}^*$) is sequentially weak*-compact.

Hint. Let $\{x_i\}$ be a countable dense subset of $\mathcal{X}$. Consider any sequence of functionals $\{f_n\} \subseteq B^*$. Find a subsequence $n_k \to \infty$ and a linear functional f defined on the linear span of the x_i such that $\lim_{k\to\infty} f_{n_k}(x_i) = f(x_i)$ for all i.

Solution

The sequence $\{f_n(x_1)\}$ is bounded in $\mathbb{R}$ since $\|f_n\|_* \leqslant 1$ and

$$|f_n(x_1)| \leqslant \|x_1\| \, \|f_n\|_* \quad \text{for all } n \in \mathbb{N}.$$

Therefore there exists a subsequence $n_k \to \infty$ and a number c_1 with $|c_1| \leqslant \|x_1\|$ such that

$$\lim_{k\to\infty} f_{n_k}(x_1) = c_1.$$

We apply the same argument to the sequence $\{f_{n_k}\}$ and x_2 and find a subsequence of $\{n_k\}$ (still labeled $\{n_k\}$ for simplicity) and a number c_2 with $|c_2| \leqslant \|x_2\|$ such that

$$\lim_{k\to\infty} f_{n_k}(x_2) = c_2.$$

Repeating the same argument for all $n \in \mathbb{N}$ we obtain a subsequence of $\{n\}$ (still labeled $\{n_k\}$ for simplicity) and a sequence $\{c_i\}$ of numbers with $|c_i| \leqslant \|x_i\|$ such that

$$\lim_{k\to\infty} f_{n_k}(x_i) = c_i \quad \text{for all } i.$$

We can now define a linear functional on the span $M := \operatorname{span}(x_1, x_2, \ldots)$ by putting

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) := \alpha_1 c_1 + \alpha_2 c_2 + \ldots$$

for all numbers $\alpha_1, \alpha_2, \ldots$. This implies that

$$f(m) = \lim_{k\to\infty} f_{n_k}(m) \quad \text{for all } m \in M. \tag{5}$$

Since $\|f_{n_k}\|_* \leqslant 1$ we can estimate

$$|f(m)| = \lim_{k\to\infty} |f_{n_k}(m)| \leqslant \|m\| \quad \text{for all } m \in M,$$

so f can be extended from M to all of X by continuity. That is, for any given point $x \in X$, let $i_j \to \infty$ be a subsequence such that $x_{i_j} \to x$ and define

$$f(x) := \lim_{j\to\infty} f(x_{i_j}).$$

The existence of such a subsequence follows from the denseness of $\{x_i\}$ in X. Then

$$|f(x)| = \lim_{j\to\infty} |f(x_{i_j})| \leqslant \lim_{j\to\infty} \|x_{i_j}\| = \|x\|,$$

which implies that $\|f\|_* \leqslant 1$. For given $\varepsilon > 0$ choose x_i with $\|x - x_i\| \leqslant \varepsilon$. Then

$$\begin{aligned} |f(x) - f_{n_k}(x)| &\leqslant |f(x_i) - f_{n_k}(x_i)| + |f(x - x_i) - f_{n_k}(x - x_i)| \\ &\leqslant |f(x_i) - f_{n_k}(x_i)| + 2\|x - x_i\|, \end{aligned}$$

by linearity. As $k \to \infty$, the first term on the right-hand side converges to zero because of (5). Since $\varepsilon > 0$ was arbitrary, we therefore obtain that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \quad \text{for all } x \in X. \qquad \square$$

4. Suppose that $f_n \in C^1[0,1]$ for $n \in \mathbb{N}$, and we have:

(a) $f_n(0) = 0$,

(b) $|f_n'(x)| \leq \frac{1}{\sqrt{x}}$ a.e., and

(c) There exists a measurable function h such that $f_n'(x) \to h(x)$ for every $x \in [0,1]$.

Prove there exists an absolutely continuous function f such that f_n converges to f uniformly as $n \to \infty$.

Solution

Each function f_n is absolutely continuous, so we have

$$\int_0^x f_n' = f_n(x) - f_n(0) = f_n(x), \quad x \in [0,1].$$

Since the function $x^{-1/2}$ is integrable on $[0,1]$, we have that $h \in L^1[0,1]$. Therefore, the function

$$f(x) = \int_0^x h, \quad x \in [0,1],$$

is well-defined and is absolutely continuous on $[0,1]$. Further, $h - f_n' \to 0$ and $|h - f_n'| \leq 2x^{-1/2} \in L^1[0,1]$, so by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 |h - f_n'| = 0.$$

Hence,

$$\begin{aligned}
\sup_x |f(x) - f_n(x)| &= \sup_x \left| \int_0^x (h - f_n') \right| \\
&\leq \sup_x \int_0^x |h - f_n'| \\
&\leq \int_0^1 |h - f_n'| \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

so $f_n \to f$ uniformly. $\square$

5. Let $\mathcal{X}$ be a reflexive normed space and let $\mathcal{X}^*$ be its dual.

(a) Prove that for any sequence $x_n \rightharpoonup x$ weakly, we have
$$\|x\| \leqslant \liminf_{n\to\infty} \|x_n\|.$$

(b) Let $K \subseteq \mathcal{X}$ be a closed, nonempty, and convex set. Prove that for all $x_0 \in \mathcal{X}$ there exists $x \in K$ such that
$$\|x - x_0\| = \inf\Big\{\|y - x_0\| \colon y \in K\Big\}. \tag{6}$$

Solution

(a) Weakly convergent sequences are strongly bounded (uniform boundedness principle), therefore $\{\|x_n\|\}$ is bounded. Let $f \in \mathcal{X}^*$ with $\|f\|_* = 1$ (dual norm). Then
$$|f(x_n)| \leqslant \|f\|_* \, \|x_n\| \leqslant \|x_n\| \quad \text{for all } n \in \mathbb{N}.$$
This implies that
$$|f(x)| = \lim_{n\to\infty} |f(x_n)| \leqslant \liminf_{n\to\infty} \|x_n\|. \tag{7}$$
As a consequence of the Hahn-Banach theorem, we have the following characterization
$$\|x\| = \sup\Big\{|f(x)| \colon f \in \mathcal{X}^*, \|f\|_* = 1\Big\}.$$
Since the right-hand side of (7) does not depend on f, we can take the supremum on both sides of (7) over all $f \in \mathcal{X}^*$ with $\|f\|_* = 1$. This proves the claim.

(b) Note first that the right-hand side of (6) is finite for all $x_0 \in \mathcal{X}$ since the norm $\|\cdot\|$ is nonnegative (thus bounded below) and the set K is nonempty. If $x_0 \in K$, then we can choose $x := x_0$ and obtain identity (6) with both sides equal to zero. If $x_0 \notin K$, let L denote the right-hand side of (6). Then we consider a sequence $y_n \in K$ with
$$\lim_{n\to\infty} \|y_n - x_0\| = L. \tag{8}$$
There exists an index $N \in \mathbb{N}$ such that $\|y_n - x_0\| \leqslant L + 1$ for all $n \geqslant N$, so that
$$\|y_n\| \leqslant \|x_0\| + \|y_n - x_0\| \leqslant C \quad \text{for all } n \in \mathbb{N},$$
with C some constant. Since $\mathcal{X}$ is reflexive, the weak and the weak* topologies coincide, so we can apply the Banach-Alaoglu theorem: there exists a subsequence $n_k \to \infty$ and an element $x \in \mathcal{X}$ such that $y_n \rightharpoonup x$ weakly in $\mathcal{X}$. Since K is convex, closedness and weakly-closedness are equivalent, so $x \in K$. Moreover, because of part (a) we obtain
$$\|x - x_0\| \leqslant \liminf_{k\to\infty} \|y_{n_k} - x_0\| \overset{(8)}{=} L.$$
Since $x \in K$, we also have the reverse inequality $\|x - x_0\| \geqslant L$. □

6. Let ν be a signed Borel measure on $I = [0,1]$ such that $|\nu|(I) = 1$ and $\nu(I) = 0$. Suppose there exists a continuous function $f\colon I \to \mathbb{R}$ such that $\|f\|_\infty \leq 1$ and

$$\int_0^1 f\, d\nu = 1.$$

Show that Lebesgue measure on I is not absolutely continuous with respect to $|\nu|$.

Solution

Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν, and let $I = P \cup N$ be a corresponding Hahn decomposition of I. Then we have

$$\nu^+(P) + \nu^-(N) = |\nu|(I) = 1 \qquad \text{and} \qquad \nu^+(P) - \nu^-(N) = \nu(I) = 0,$$

so

$$\nu^+(P) = \nu^-(N) = \frac{1}{2}.$$

Also,

$$1 = \int_0^1 f\, d\nu = \int_P f\, d\nu^+ - \int_N f\, d\nu^-.$$

Since $-1 \leq f \leq 1$ everywhere,

$$\int_P f\, d\nu^+ \leq \int_P d\nu^+ = \nu^+(P) = \frac{1}{2},$$

and similarly

$$-\int_N f\, d\nu^- \leq \int_N d\nu^- = \nu^-(N) = \frac{1}{2}.$$

In other for the sum of these two integrals to be 1, we must therefore have

$$\int_P f\, d\nu^+ = -\int_N f\, d\nu^- = \frac{1}{2}.$$

Hence

$$\int_P (1-f)\, d\nu^+ = 0,$$

so since $1 - f \geq 0$ and $\nu^+ \geq 0$, we must have $f = 1$ ν-a.e. on P. Similarly, $f = -1$ ν-a.e. on N, and in particular we have $|f| = 1$ ν-a.e.

Thus f takes both the values 1 and -1. But f is continuous, so there is an open interval $U \subseteq I$ with $-1 < f(x) < 1$ for $x \in U$. However, we must have $|\nu|(U) = 0$, so since U has positive Lebesgue measure we conclude that Lebesgue measure is not absolutely continuous with respect to ν. □

7. Let $(X, \mathcal{M}, \mu)$ be a measure space (not necessarily σ-finite).

(a) Suppose that f_n, f are measurable functions from X to $[0, \infty]$. Prove that if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for all } x \in X,$$

then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

(b) Suppose $\{f_n\}$ is a sequence of measurable functions from X to $\mathbb{R}$ (not necessarily nonnegative), defined a.e., such that

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$

Prove that the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \tag{9}$$

converges for almost all $x \in X$, that $f \in L^1(\mu)$, and that

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu. \tag{10}$$

Solution

(a) Consider first the case of two functions f_1 and f_2. By measurability, there exist sequences $\{s_{1,k}\}$ and $\{s_{2,k}\}$ of nonnegative simple functions such that

$$s_{1,k}(x) \to f_1(x) \quad \text{and} \quad s_{2,k}(x) \to f_2(x) \quad \text{for all } x \in X$$

monotonically from below. Defining $t_k := s_{1,k} + s_{2,k}$ for all k, we have that

$$t_k(x) \to f_1(x) + f_2(x) \quad \text{for all } x \in X$$

monotonically from below. The Monotone Convergence Theorem then shows that

$$\begin{aligned}
\int_X (f_1 + f_2) \, d\mu &= \lim_{k\to\infty} \int_X t_k \, d\mu \\
&= \lim_{k\to\infty} \left\{ \int_X s_{1,k} \, d\mu + \int_X s_{2,k} \, d\mu \right\} \\
&= \left\{ \lim_{k\to\infty} \int_X s_{1,k} \, d\mu \right\} + \left\{ \lim_{k\to\infty} \int_X s_{2,k} \, d\mu \right\} \\
&= \int_X f_1 \, d\mu + \int_X f_2 \, d\mu. \qquad (11)
\end{aligned}$$

Next, put $g_N := f_1 + \ldots + f_N$ for all $N \in \mathbb{N}$. The sequence $\{g_N\}$ converges monotonically from below to f, because the f_n are nonnegative. Applying induction to (11), we get

$$\int_X g_N \, d\mu = \sum_{n=1}^{N} \int_X f_n \, d\mu. \tag{12}$$

Applying the monotone convergence theorem once again, we obtain from (12)

$$\int_X f \, d\mu = \lim_{N\to\infty} \int_X g_N \, d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

(b) Let S_n be the set on which f_n is defined. Then $\mu(X \setminus S_n) = 0$ for all n. Let

$$\varphi(x) := \sum_{n=1}^{\infty} |f_n(x)| \quad \text{for all } x \in S,$$

where $S := \bigcap_{n=1}^{\infty} S_n$ with $\mu(X \setminus S) = 0$. Applying part (a), we obtain

$$\int_X \varphi \, d\mu = \sum_{n=1}^{\infty} \int_X |f_n| \, d\mu,$$

which is finite by assumption. Therefore the set

$$E := \Big\{ x \in S \colon \varphi(x) < \infty \Big\}$$

satisfies $\mu(X \setminus E) = 0$, and the series (9) converges absolutely for all $x \in E$. If we define $f(x)$ by (9) for all $x \in E$, then $|f(x)| \leqslant \varphi(x)$ on E, which implies $f \in L^1(\mu)$. Moreover, letting $g_N := f_1 + \ldots + f_N$ for all $N \in \mathbb{N}$, then $|g_N| \leqslant \varphi$ and $g_N(x) \to f(x)$ for all $x \in E$. By the Dominated Convergence Theorem, we obtain

$$\int_E f \, d\mu = \lim_{N\to\infty} \int_E g_N \, d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_E f_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

Since $\mu(X \setminus E) = 0$, this is equivalent to (10). □

8. Fix $1 < p < \infty$, and let p' satisfy $\frac{1}{p} + \frac{1}{p'} = 1$. Let $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$ be given. Show that the function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

exists and belongs to $C_0(\mathbb{R})$, i.e., $f * g$ is continuous, and satisfies

$$\lim_{x \to \pm\infty} (f * g)(x) = 0.$$

Solution

Given any fixed x, we have by Hölder's inequality that the function $f(\cdot)\, g(x - \cdot)$ is integrable. Hence $f * g$ is well-defined at every point, and is bounded since

$$|(f * g)(x)| \le \int |f(t)\, g(x-t)|\, dt \le \left(\int |f(t)|^p\, dt \right)^{1/p} \left(\int |g(x-t)|^{p'}\, dt \right)^{1/p'} = \|f\|_p\, \|g\|_{p'}.$$

Thus, we actually have

$$\|f * g\|_\infty \le \|f\|_p\, \|g\|_{p'}.$$

Again fix $x \in \mathbb{R}$. Then given any $h \in \mathbb{R}$, if we set $T_h g(x) = g(x-h)$ then we can write

$$\begin{aligned}
|(f * g)(x+h) - (f * g)(x)| &\le \int |f(t)|\, |g(x+h-t) - g(x-t)|\, dt \\
&\le \left(\int |f(t)|^p\, dt \right)^{1/p} \left(\int |g(x+h-t) - g(x-t)|^{p'}\, dt \right)^{1/p'} \\
&= \|f\|_p \left(\int |g(t-h) - g(t)|^{p'}\, dt \right)^{1/p'} \\
&= \|f\|_p\, \|T_h g - g\|_{p'} \\
&\to 0 \quad \text{as } h \to 0,
\end{aligned}$$

the convergence following from the fact that translation is a strongly continuous family of operators on $L^{p'}(\mathbb{R})$ (this statement can be proved by using an approximation argument similar to the one we use next).

Finally, since $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ and in $L^{p'}(\mathbb{R})$, we can find continuous, compactly supported functions f_n, g_n such that $\|f - f_n\|_p \to 0$ and $\|g - g_n\|_p \to 0$. As above, $f_n * g_n$ is continuous, and furthermore it is compactly supported, since

$$\mathrm{supp}(f_n * g_n) \subseteq \mathrm{supp}(f_n) + \mathrm{supp}(g_n).$$

Thus $f_n * g_n \in C_c(\mathbb{R}) \subseteq C_0(\mathbb{R})$ for each n. Further, $\sup \|f_n\|_p < \infty$, so

$$\begin{aligned}
\|f * g - f_n * g_n\|_\infty &\le \|f * g - f_n * g\|_\infty + \|f_n * g - f_n * g_n\|_\infty \\
&\le \|f - f_n\|_p\, \|g\|_{p'} + \|f_n\|_p\, \|g - g_n\|_{p'} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus $f_n * g_n \to f * g$ uniformly. But $C_0(\mathbb{R})$ is a Banach space with respect to the uniform norm, so this implies that $f * g \in C_0(\mathbb{R})$. $\square$