Analysis Comprehensive Exam Questions
Spring 2009

Problem 1. Let $X := (0, \infty)$ and $1 < p < \infty$. For any $f \in \mathscr{L}^p(X, m)$, where m is the Lebesgue measure, define the function

$$F(x) := \frac{1}{x} \int_0^x f(t)\, dt \quad \text{for all } x \in X.$$

Then prove Hardy's inequality

$$\|F\| \leqslant \frac{p}{p-1} \|f\|.$$

Hint: Assume first that $f \in C_c(X)$ and $f \geqslant 0$, *i.e.* f is a continuous and positive function with compact support in X. Integration by parts gives

$$\int_0^\infty F^p(x)\, dx = -p \int_0^\infty F^{p-1}(x) x F'(x)\, dx.$$

Note that $xF'(x) = f(x) - F(x)$.

Solution. Observe first that, since $f \in \mathscr{L}^p(X, m)$, $f \in \mathscr{L}^1((0, x], m)$ for every $x \in X$. Thus $F(x)$ is well defined.
Let $f \in C_c(X)$ and $f \geqslant 0$. Then there exists a number $\delta > 0$ such that $\operatorname{spt} f \subset [\delta, \delta^{-1}]$. This implies that $F(x) = 0$ if $x \leqslant \delta$ and $F(x) = C/x$ for $x \geqslant \delta^{-1}$, with $C > 0$ some constant. Hence $F(x) \in \mathscr{L}^p(X, m)$.
Since f is continuous, the function F is continuously differentiable, by the fundamental theorem of calculus. Integration by parts then gives

$$\int_0^\infty F^p(x)\, dx = -p \int_0^\infty F^{p-1}(x) x F'(x)\, dx = -p \int_0^\infty F^{p-1}(x) f(x)\, dx + p \int_0^\infty F^p(x)\, dx,$$

since $xF'(x) = f(x) - F(x)$. We used that F vanishes at the boundary. Then

$$\int_0^\infty F^p(x)\, dx \leqslant \frac{p}{p-1} \int_0^\infty F^{p-1}(x) f(x)\, dx.$$

Applying Hölder inequality to the right hand side, we obtain

$$\int_0^\infty F^p(x)\, dx \leqslant \frac{p}{p-1} \left(\int_0^\infty F^p(x)\, dx \right)^{(p-1)/p} \left(\int_0^\infty f^p(x)\, dx \right)^{1/p}.$$

Then Hardy's inequality follows because p and $p/(p-1)$ are conjugate exponents.
For general nonnegative $f \in \mathscr{L}^p(X, m)$ consider a sequence of nonnegative functions $f_n \in C_c(X)$ with $f_n \longrightarrow f$ a.e. monotonically.
By the monotone convergence theorem, we then have

$$F_n(x) := \frac{1}{x} \int_0^x f_n(t)\, dt \longrightarrow \frac{1}{x} \int_0^x f(t)\, dt = F(x)$$

for all $x \in X$ monotonically.
Hence $F_n^p(x) \longrightarrow F^p(x)$ and $f_n^p(x) \longrightarrow f^p(x)$ monotonically.
Applying the monotone convergence theorem again, we conclude that

$$\|F\| = \lim_{n \to \infty} \|F_n\| \leqslant \frac{p}{p-1} \lim_{n \to \infty} \|f_n\| = \frac{p}{p-1} \|f\|.$$

If f is real- or complex-valued, then we use that

$$|F(x)| = \left| \frac{1}{x} \int_0^x f(t)\, dt \right| \leqslant \frac{1}{x} \int_0^x |f(t)|\, dt \quad \text{for all } x \in X.$$

Applying Hardy's inequality to $|f|$ instead of f we obtain

$$\|F\| \leqslant \left\| \frac{1}{x} \int_0^x |f(t)|\, dt \right\| \leqslant \frac{p}{p-1} \| \, |f| \, \| = \frac{p}{p-1} \|f\|.$$

Problem 2. Let $X := [0,1]$.

(1) For any $n \geqslant 2$ let A_n denote the set of all functions $f \in C(X)$ for which there exists a point $x \in [0, 1-\frac{1}{n}]$ such that $|f(x+h) - f(x)| \leqslant nh$ for all $0 < h < \frac{1}{n}$. Show that A_n is nowhere dense in $C(X)$ with the uniform topology.

(2) Use the above result to show that the set of functions on X that do not admit left nor right derivatives at any point of X is dense in $C(X)$ in the uniform topology.

Solution. For any $n \geqslant 2$ let A_n be defined as above. We prove that A_n is closed. Let $\{f_k\} \subset A_n$ such that $\lim_{k\to\infty} \|f - f_k\|_\infty = 0$. For every k there exists x_k such that $|f_k(x_k+h) - f_k(x_k)| \leqslant nh$ for all $0 < h < \frac{1}{n}$. Since $[0, 1-\frac{1}{n}]$ is compact there exists a subsequence x_{k_j} that converges to a point $x \in [0, 1-\frac{1}{n}]$. We then have that $\lim_{j\to\infty} f_{k_j}(x_{k_j}+h) = f(x+h)$ for every h, hence

$$|f(x+h) - f(x)| \leqslant nh \quad \text{for all } 0 < h < \frac{1}{n},$$

and so $f \in A_n$.

We now prove that A_n is nowhere dense in $C(X)$, by showing that in any ε-neighborhood around any function $f \in A_n$, there exists $\tilde{f} \in C(X) \setminus A_n$. Hence the interior of A_n is empty.

To this end, let $g \in C(X)$ be such that $|g(x)| \leq \varepsilon$ and for every x we have $|g(x+h) - g(x)| \geq 3hn$ for $h > 0$ small enough (depending on x). For example, let

$$g(x) := \begin{cases} 3nx & 0 \leqslant x \leqslant \frac{\varepsilon}{3n}, \\ -3nx + 2\varepsilon & \frac{\varepsilon}{3n} \leqslant x \leqslant \frac{2\varepsilon}{3n}, \end{cases}$$

and $g(x) := g(x - \frac{2\varepsilon}{3n})$ for $x \notin [0, \frac{2\varepsilon}{3n}]$.

Given $f \in A_n$, we define $\tilde{f} := f + g$, which implies that $\|\tilde{f} - f\|_\infty = \|g\|_\infty = \varepsilon$.

But for every $x \in [0, 1-\frac{1}{n}]$ and h small enough we have

$$|\tilde{f}(x+h) - \tilde{f}(x)| \geqslant |g(x+h) - g(x)| - |f(x+h) - f(x)| \geqslant 3nh - nh > nh,$$

so that $\tilde{f} \notin A_n$.

From the above result it follows that for every n the set A_n^c is open and dense in $C(X)$. From Baire's category theorem we know that $A = \bigcap_{n=2}^\infty A_n^c$ is dense in $C(X)$. Observe that if $f \in A$, then for every $x \in X$ and every $n \geqslant 2$ there exists a $0 < h < \frac{1}{n}$ such that

$$\left| \frac{f(x+h) - f(x)}{h} \right| > n,$$

so that the right derivative of f at x does not exists.

In a similar fashion, we let B_n denote the set of all functions $f \in C(X)$ for which there exists a point $x \in [\frac{1}{n}, 1]$ such that $|f(x-h) - f(x)| \leqslant nh$ for all $0 < h < \frac{1}{n}$. Then one can show that B_n is closed and nowhere dense, for all $n \geqslant 2$. Defining $B := \bigcap_{n=2}^\infty B_n^c$, we use again Baire's category theorem to conclude that B is dense in $C(X)$. If $f \in B$, then for every $x \in X$ the left derivative of f at x does not exists.

Finally the set $C := A \cup B$ is dense and if $f \in C$ then for every $x \in X$ the left and right derivatives of f at x do not exist.

Problem 3. Let B be a Banach space.

(1) Prove that if $T\colon X \longrightarrow X$ is a bounded linear operator with $\|\mathrm{id} - T\| < 1$, where id is the identity operator on X, then T is invertible.

(2) Let T be as before and consider another bounded linear operator $S\colon X \longrightarrow X$ with $\|S - T\| < \|T^{-1}\|^{-1}$. Prove that S is invertible. Show that the set of invertible bounded linear operators from X to itself is an open set in the operator norm topology.

Solution. We write $T = \mathrm{id} - (\mathrm{id} - T)$ and claim that

$$(\mathrm{id} - (\mathrm{id} - T))^{-1} = \sum_{k=0}^{\infty} (\mathrm{id} - T)^k.$$

We note first that the series is absolutely convergent because

$$\sum_{k=0}^{\infty} \|(\mathrm{id} - T)^k\| \leqslant \sum_{k=0}^{\infty} \|\mathrm{id} - T\|^k,$$

which is finite because $\|\mathrm{id} - T\| < 1$ and so the series is a geometric series.
Absolute convergence implies convergence in norm.
For any $N \in \mathbb{N}$ we can now write

$$(\mathrm{id} - (\mathrm{id} - T)) \circ \sum_{k=0}^{N} (\mathrm{id} - T)^k = \mathrm{id} - (\mathrm{id} - T)^{N+1}.$$

Sending $N \to \infty$ and using the fact that

$$\|(\mathrm{id} - T)^{N+1}\| \leqslant \|\mathrm{id} - T\|^{N+1} \longrightarrow 0,$$

again because $\|\mathrm{id} - T\| < 1$, we prove the claim.
Note now that

$$\|S \circ T^{-1} - \mathrm{id}\| = \|(S - T) \circ T^{-1}\| \leqslant \|S - T\| \|T^{-1}\| < 1,$$

by assumption. Applying (1), we find that the operator $S \circ T^{-1}$ is invertible, so

$$(S \circ T^{-1})^{-1} = T \circ S^{-1}$$

exists. Since T is invertible by assumption, we conclude that S is invertible as well and

$$S^{-1} = T^{-1} \circ (S \circ T^{-1})^{-1}.$$

This shows that the set of invertible operators is open because for any invertible operator T, the open ball around T with radius less than $\|T^{-1}\|^{-1}$ is also contained in the set of invertible operators.

Problem 4. Let X be a compact metric space and $\mathcal{A}$ a closed algebra of continuous real-valued functions separating points in X.

(1) Show that if $f \in \mathcal{A}$ then $\sqrt{|f|} \in \mathcal{A}$.
(2) Show that $\mathcal{A}$ is either all of $C(X)$ or there exists a point $x \in X$ such that

$$\mathcal{A} = \{f \in C(X) \mid f(x) = 0\}.$$

Hint: Prove first that there exists a sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$ such that $P_n(z) \longrightarrow \sqrt{z}$ uniformly on $[\varepsilon, 1]$ for any $\varepsilon > 0$. To this end, consider the iteration

$$P_1(z) := 0 \quad \text{and} \quad P_{n+1}(z) := P_n(z) + \frac{1}{2}\big(z - P_n(z)^2\big) \quad \text{for all } z \in [0,1], n \in \mathbb{N}.$$

Solution. Let the polynomials P_n be defined as above and note that by induction we have that $P_n(0) = 0$ for all $n \in \mathbb{N}$. Observe now that for all $z \in [0,1]$ it holds

$$z - P_{n+1}(z)^2 = \big(z - P_n(z)^2\big)\left(1 - P_n(z) - \frac{1}{4}\big(z - P_n(z)^2\big)\right)$$

and that if $0 \le P_n(z) \le \sqrt{z}$ then

$$0 \le 1 - \sqrt{z} \le 1 - P_n(z) - \frac{1}{4}\left(z - P_n(z)^2\right) \le 1 - \frac{1}{4}z$$

This easily implyes that $z - P_{n+1}(z)^2 \ge 0$ (since it is the product of two positive quantities) and thus $P_{n+1}(z) \ge P_n(z) \ge 0$. By induction again we have that $P_n(z)$ is an increasing sequence with $0 \le P_n(z) \le \sqrt{z}$ for every n and that $P_n(z)$ converges uniformly to $\sqrt{z}$ in $C([\epsilon, 1])$ for every ϵ. Since $\sqrt{z}$ and $P_n(z)$ are continuous functions the convergence is uniform in $C([0,1])$.
Given $f \in \mathcal{A}$ set $c = \|f\|_\infty$. We have that $P_n(f^2/c^2) \in \mathcal{A}$ for every n. Moreover, since $0 \le f^2/c^2 \le 1$, $P_n(f^2/c^2)$ converges uniformly to $|f|/c$. Hence $|f| \in \mathcal{A}$. By the same argument we get that $P_n(|f|/c)$ converges uniformly to $\sqrt{|f|/c}$ and thus $\sqrt{|f|} \in \mathcal{A}$.

(2) Suppose that there is no $x \in X$ such that $f(x) = 0$ for all $f \in \mathcal{A}$. Thus for every $x \in X$ we can find $f_x \in \mathcal{A}$ such that $f_x(x) \neq 0$. Since f_x is continuous there exists an open neighborhood U_x of x such that $f_x(y) \neq 0$ for all $y \in U_x$.
Observe that $\{U_x\}_{x\in X}$ is an open covering of X. Since X is compact, there exists a finite sub-covering U_{x_i}, $i = 1, \dots, n$. Consider now the function $g = \sum_{i=1}^n f_{x_i}^2$. Clearly $g \in \mathcal{A}$ and $g(x) > 0$ for all $x \in X$. Dividing g by its norm, we may assume that $g(x) \leqslant 1$ for every $x \in X$. Moreover, since X is compact there exists a constant $\varepsilon > 0$ with $\varepsilon \leqslant g(x)$ for all $x \in X$.
We now define the functions $h_N(x) := \sqrt[2^N]{g(x)}$ for all $x \in X$ and $N \in \mathbb{N}$, which is in $\mathcal{A}$ by (1). We have that $\lim_{n\to\infty} h_N(x) = 1$ for every $x \in X$. In fact, the convergence is uniformly because $g(x) \geqslant \varepsilon > 0$. Since $\mathcal{A}$ is closed, we conclude that the constant functions are in $\mathcal{A}$.
Thus $\mathcal{A}$ is an algebra of continuous function that separates points and contains the constant functions. Therefore, by the Stone-Weierstrass theorem, we have that $\mathcal{A} = C(X)$.
On the other hand, if there exists $x \in X$ such that $f(x) = 0$ for all $f \in \mathcal{A}$, then for every $y \neq x$ there exists $f_y \in \mathcal{A}$ with $f_y(y) \neq 0$ because $\mathcal{A}$ separates points. That is, there can at most exist one point in X at which all functions in $\mathcal{A}$ vanish. The same argument as before then implies that $\mathcal{A}$ contains all continuous functions f with $f(x) = 0$.

Problem 5. Let X be some set and $\mathscr{P}(X)$ its power set. Consider a map $K\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ with the following properties:

(1) $K(\varnothing) = \varnothing$;
(2) $A \subset K(A)$ for all A;
(3) $K(K(A)) = K(A)$ for all A;
(4) $K(A \cup B) = K(A) \cup K(B)$ for all A, B,

and let $\mathscr{F} := \{A \subset X : K(A) = A\}$. Prove that $\varnothing, X \in \mathscr{F}$, and that $\mathscr{F}$ is closed under arbitrary intersections and finite unions. It follows that $\mathscr{T} := \{U \subset X : U^c \in \mathscr{F}\}$ is a topology. Prove that for every A, the set $K(A)$ is the closure of A with respect to the topology $\mathscr{T}$.

Solution. Since $A \subset B$ we have $B = A \cup B$. Then (4) implies

$$K(B) = K(A \cup B) = K(A) \cup K(B) \supset K(A).$$

By (1), we have $K(\varnothing) = \varnothing$ and thus $\varnothing \in \mathscr{F}$.
By (2), we have $X \subset K(X)$ and clearly $K(X) \subset X$. Hence $X \in \mathscr{F}$.
Let $A_\lambda \in \mathscr{F}$, $\lambda \in \Lambda$, be an arbitrary collection of sets. Then $\bigcap_{\lambda\in\Lambda} A_\lambda \subset A_\beta$ for all β. Therefore

$$K\Bigg(\bigcap_{\lambda\in\Lambda} A_\lambda\Bigg) \subset K(A_\beta)$$

for all β, and so

$$K\Bigg(\bigcap_{\lambda\in\Lambda} A_\lambda\Bigg) \subset \bigcap_{\beta\in\Lambda} K(A_\beta) = \bigcap_{\beta\in\Lambda} A_\beta \subset K\Bigg(\bigcap_{\beta\in\Lambda} A_\beta\Bigg).$$

The equality follows from the assumption $K(A_\beta) = A_\beta$ for all β, and the last inclusion from (2). We conclude that all terms are in fact equal, hence $\mathscr{F}$ is closed under arbitrary intersections.
Let now $A_1, A_2 \in \mathscr{F}$ be given. From (4) we obtain

$$K(A_1 \cup A_2) = K(A_1) \cup K(A_2) = A_1 \cup A_2.$$

The last equality follows from the assumption that $K(A_i) = A_i$ for $i = 1, 2$. By induction, we have that $\mathscr{F}$ is closed under finite unions.
The collection $\mathscr{F}$ contains all closed sets of the topology $\mathscr{T}$ because B being closed is, by definition, equivalent to B^c being open, that is $B^c \in \mathscr{T}$. This in turn is equivalent to $(B^c)^c = B \in \mathscr{F}$.
Note that for any A, the set $K(A) \in \mathscr{F}$ because of (3).
Let now $\bar{A} := \bigcap\{B \in \mathscr{F} : A \subset B\}$ be the closure of A in the topology $\mathscr{T}$.
Since $A \subset K(A)$ by (2), and since $K(A) \in \mathscr{F}$, we have $\bar{A} \subset K(A)$.
On the other hand, since $\mathscr{F}$ is closed under arbitrary intersections, we have $\bar{A} \in \mathscr{F}$. Then $A \subset \bar{A}$ implies

$$K(A) \subset K(\bar{A}) = \bar{A} \subset K(A),$$

and so all sets are in fact equal.

Problem 6. Let $(X, \mathscr{M}, \mu)$ be a measure space and consider functions $f \in L^p(\mu)$ and $g \in L^q(\mu)$ with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that $\int_X |fg|\, d\mu = \|f\|_p \|g\|_q$ if and only if there exist constants $C_1, C_2 \geqslant 0$, not both equal to zero, such that $C_1|f|^p = C_2|g|^q$.
Hint: Show first that for all $a, b \geqslant 0$ and $t \in (0,1)$ we have $a^t b^{1-t} \leqslant ta + (1-t)b$, with equality holding if and only if $b = a$. To this end, consider $h(x) := 1 - t + tx - x^t$ for $x \in [0,1]$.

Solution. Let h be defined as above and note that $h'(x) = t(1 - x^{t-1}) < 0$ for all $x \in [0,1]$ and $t \in (0,1)$. Since $h(1) = 0$ we conclude that $h(x) > 0$ for all $x \in [0,1)$. With $x := a/b$ we find

$$a^t b^{1-t} \leqslant ta + (1-t)b, \tag{1}$$

and equality holds if and only if $a = b$. If $b = 0$, then there is nothing to prove.
Assume now that $C_1|f|^p = C_2|g|^q$ with $C_1 > 0$. We have

$$\begin{aligned}\int_X |fg|\, d\mu = \left(\frac{C_2}{C_1}\right)^{\frac{1}{p}} \int_X |g|^q\, d\mu &= \left[\int_X \left(\frac{C_2}{C_1}\right) |g|^q\, d\mu\right]^{\frac{1}{p}} \left[\int_X |g|^q\, d\mu\right]^{\frac{1}{q}} \\ &= \left[\int_X |f|^p\, d\mu\right]^{\frac{1}{p}} \left[\int_X |g|^q\, d\mu\right]^{\frac{1}{q}} = \|f\|_p \|g\|_q.\end{aligned}$$

If $C_1 = 0$, then $g = 0$ and the identity follows trivially.
For the converse direction, assume that $\int_X |fg|\, d\mu = \|f\|_p \|g\|_q$. This identity is certainly satisfied if $f = 0$, in which case we may take $C_1 = 1$ and $C_2 = 0$ (and thus g is arbitrary). In a similar way, we may argue if $g = 0$.
Assume therefore that neither f or g are the zero function. Now fix $x \in X$ and let

$$t := \frac{1}{p}, \quad a := \left(\frac{|f(x)|}{\|f\|_p}\right)^p, \quad b := \left(\frac{|g(x)|}{\|g\|_q}\right)^q,$$

which are well-defined since $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. From inequality (1) we obtain

$$\frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q - \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leqslant 0.$$

Integrating we get

$$\int_X \left[\frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q - \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}\right] d\mu = \frac{1}{p} + \frac{1}{q} - \frac{\int_X |f(x)g(x)|\, d\mu}{\|f\|_p\|g\|_q} = 0,$$

where the last equality follows from our hypothesis. This implies that

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} = \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

for μ-a.e. $x \in X$. Since equality holds in (1) if and only if $a = b$ we get that

$$\left(\frac{|f(x)|}{\|f\|_p}\right)^p = \left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

for μ-a.e. $x \in X$. It is now enough to take $C_1 := \|g\|_q^q$ and $C_2 := \|f\|_p^p$ to obtain the thesis.

Problem 7. Let $(X, \mathscr{M}, \mu)$ be a measure space. For any given set $E \in \mathscr{M}$ we denote by $\mathscr{L}^2(E,\mu)$ the subspace of $\mathscr{L}^2(X,\mu)$ of functions that vanish in $X \setminus E$. Let $\{E_n\}$ be a sequence of pairwise disjoint sets $E_n \in \mathscr{M}$ with $X = \bigcup_{n=1}^\infty E_n$.
Prove that $\{\mathscr{L}^2(E_n,\mu)\}$ is a sequence of mutually orthogonal subspaces of $\mathscr{L}^2(X,\mu)$, and that every $f \in \mathscr{L}^2(X,\mu)$ can be written uniquely as $f = \sum_{n=1}^\infty f_n$ with $f_n \in \mathscr{L}^2(E_n,\mu)$ (prove that the series converges in norm).

Solution. Let $f_m \in \mathscr{L}^2(E_m,\mu)$ and $f_n \in \mathscr{L}^2(E_n,\mu)$ with $m \neq n$. Since $E_m \cap E_n = \varnothing$, we find that f_m vanishes on $E_n \subset X \setminus E_m^c$ and f_n vanishes on $E_m \subset X \setminus E_n^c$. Therefore we have

$$\int_X f_m f_n \, d\mu = 0,$$

so the two subspaces are mutually orthogonal with respect to the inner product.
Let now $f \in \mathscr{L}^2(X,\mu)$ be given and define

$$f_n := f\chi_{E_n} \in \mathscr{L}^2(E_n,\mu) \quad \text{for all } n \in \mathbb{N}.$$

Then the functions f_n are mutually orthogonal as shown above.
Since $f \in \mathscr{L}^2(X,\mu)$, we have $|f|^2 \in \mathscr{L}^1(X,\mu)$, and so $|f|^2\mu$ is a finite measure.
Define $F_N := \bigcup_{n=1}^N E_n$ for all $N \in \mathbb{N}$. Then $\{F_N\}$ is a monotone increasing sequence of measurable sets, with $X = \bigcup_{N=1}^\infty F_N$.
For any N we now write

$$\sum_{n=1}^N \int_X |f_n|^2 \, d\mu = \sum_{n=1}^N \int_{E_n} |f|^2 \, d\mu = \int_{F_N} |f|^2 \, d\mu.$$

Sending $N \to \infty$ and using continuity from below, we obtain

$$\sum_{n=1}^\infty \int_X |f_n|^2 \, d\mu = \lim_{N\to\infty} \int_{F_N} |f|^2 \, d\mu = \int_X |f|^2 \, d\mu,$$

which is finite.
Let now $M, N \in \mathbb{N}$ be given an assume without loss of generality that $N \leqslant M$. Then

$$\left\| \sum_{n=1}^M f_n - \sum_{n=1}^N f_n \right\|^2 = \left\| \sum_{n=N+1}^M f_n \right\|^2 = \int_X \left| \sum_{n=N+1}^M f_n \right|^2 d\mu = \sum_{n=N+1}^M \int_X |f_n|^2 \, d\mu,$$

which converges to zero as $M, N \to \infty$. In the last equality we used the orthogonality of the f_n.
Using the Cauchy criterion, we conclude that the series $\sum_{n=1}^\infty f_n$ converges in norm.
Multiplying the identity $f = \sum_{n=1}^\infty f_n$ by χ_{E_m} for some $m \in \mathbb{N}$, we find

$$f\chi_{E_m} = \left(\sum_{n=1}^\infty f_n \right) \chi_{E_m} = f_m \chi_{E_m} = f_m,$$

so that is the only way to define the functions f_m.

Problem 8. Let $(X, \mathscr{M}, \mu)$ be a measure space. Show that $f : X \longrightarrow [0, \infty)$ is measurable if and only if there exist nonnegative constants $\{c_n\}_{n=0}^{\infty}$ and measurable sets $\{E_n\}_{n=0}^{\infty}$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n \chi_{E_n}(x). \tag{2}$$

Solution. If (2) holds, then $f(x) = \lim_{N \to \infty} f_N(x)$, with functions f_N defined by

$$f_N(x) := \sum_{n=0}^{N} c_n \chi_{E_n}(x) \quad \text{for all } x \in X.$$

The function f is therefore the pointwise limit of the sequence measurable functions f_N, and so f is measurable as well.

For the converse direction, we use that for any measurable function $g \colon X \longrightarrow [0, \infty)$ and any $N \in \mathbb{N}$, the simple function

$$\phi(x) := \sum_{n=0}^{2^{2N}-1} \frac{n}{2^N} \chi_{F_n}, \quad \text{where} \quad F_n := g^{-1}\left(\left[\frac{n}{2^N}, \frac{n+1}{2^N} \right) \right)$$

satisfies $0 \leqslant \phi(x) \leqslant g(x)$ for all $x \in X$ and $g(x) - \phi(x) \leqslant 2^{-N}$ for all $x \in g^{-1}([0, 2^N))$.

Starting from our function f we construct the simple function ϕ_1 such that $0 \leqslant f(x) - \phi_1(x) \leqslant 2^{-1}$ for all $x \in f^{-1}([0, 2^1))$.

Now observe that the function $f_1 := f - \phi_1$ is again nonnegative and measurable, so we can find a simple function ϕ_2 with $0 \leqslant f_1(x) - \phi_2(x) \leqslant 2^{-2}$ for all $x \in f_1^{-1}([0, 2^2))$.

We define recursively a sequence of simple functions ϕ_n and measurable functions f_n such that

(1) $f_0 := f$;
(2) ϕ_n is such that $0 \leqslant f_{n-1}(x) - \phi_n(x) \leqslant 2^{-n}$ for every $x \in f_{n-1}^{-1}([0, 2^n))$;
(3) $f_n := f_{n-1} - \phi_n$.

Since $0 \leqslant f_n(x) \leqslant f(x)$ we have that for every $x \in f^{-1}([0, 2^n))$ it holds

$$0 \leqslant f(x) - \sum_{i=0}^{n} \phi_i(x) \leqslant 2^{-n}.$$

Since the function f takes only finite values, this implies that for all $x \in X$ we have

$$f(x) = \sum_{n=0}^{\infty} \phi_n(x).$$

If we now write

$$\phi_n(x) =: \sum_{i=0}^{N_n} c_{n,i} \chi_{E_{n,i}}$$

for suitable nonnegative constants $c_{n,i}$ and measurable sets $E_{n,i}$, then we obtain that

$$f(x) = \sum_{n=0}^{\infty} \sum_{i=0}^{N_n} c_{n,i} \chi_{E_{n,i}}.$$

The thesis follows then by renaming the sets $E_{n,i}$ and the numbers $c_{n,i}$.