Qualification Problems
Real Analysis

1. Let (X, d) be a metric space. Suppose that K is a compact subset of X and F is a closed subset of X. Assume $K \cap F = \emptyset$. Show that there exists a $\delta > 0$ such that

$$d(x, y) \geq \delta > 0, \quad x \in K, y \in F.$$

Solution:
Solution One: For every $x \in X$ define the distance from x to F by

$$d(x, F) = \inf\{d(x, f) : f \in F\}.$$

If $d(x, F) = 0$, then for every $\epsilon > 0$, there exists a $y_\epsilon \in F$ such that $d(x, y_\epsilon) < \epsilon$. This implies that x is a limit point of F, and since F is closed, we must have $x \in F$. Thus, $d(x, F) = 0$ if and only if $x \in F$.

Now define the following function $f : K \to [0, \infty)$ by $f(x) = d(x, F)$. It is easy to show that f is continuous. Since K is compact, there exists a minimum value attained at some value $k \in K$ with $f(k) = \delta$. Since $K \cap F = \emptyset$ and by above we must have $\delta > 0$. But, since δ is the minimum for all $x \in K$ we have that

$$f(x) = d(x, F) \geq \delta \quad x \in K.$$

But, this also gives that

$$d(x, y) \geq \delta > 0 \quad x \in K, y \in F.$$

Solution Two: Let $B(z, r)$ denote the open ball with center $z \in X$ and radius $r > 0$. As F is closed and $F \cap K = \emptyset$, for every $x \in K$ there is a radius $r(x) > 0$ such that $F \cap B(x, r(x)) = \emptyset$. Consider now the cover of K given by $\left\{B(x, \frac{r(x)}{2}) : x \in K\right\}$. As K is compact, there are points $x_1, \dots x_N$ such that $B\left(x_j, \frac{r(x_j)}{2}\right)$ $j = 1, \dots N$ is a cover of K.

Let $\delta = \min\left\{\frac{r(x_j)}{2} : j = 1, \dots N\right\}$. Then note that for each $x \in K$ there exists a x_j such that $d(x, x_j) \leq \frac{r(x_j)}{2}$. And, for each $y \in F$ we have that $d(y, x_j) > 2\frac{r(x_j)}{2}$. Now we then have that $d(x, y) \geq \delta$ for $x \in K$ and $y \in F$. Indeed,

$$\begin{aligned} d(x, y) &\geq d(x_j, y) - d(x, x_j) \\ &\geq 2\frac{r(x_j)}{2} - \frac{r(x_j)}{2} \\ &\geq \frac{r(x_j)}{2} \geq \delta > 0. \end{aligned}$$

This then gives the result.

2. Given $2 \leq p < \infty$, show that for real-valued functions $f, g \in L^p(E;\mu)$ we have

$$2\left(\left\|\frac{f}{2}\right\|^p_{L^p(E;\mu)} + \left\|\frac{g}{2}\right\|^p_{L^p(E;\mu)}\right) \leq \left\|\frac{f-g}{2}\right\|^p_{L^p(E;\mu)} + \left\|\frac{f+g}{2}\right\|^p_{L^p(E;\mu)} \leq \frac{1}{2}\left(\|f\|^p_{L^p(E;\mu)} + \|g\|^p_{L^p(E;\mu)}\right).$$

Solution:
For the upper inequality, prove the following: Let $2 \leq p < \infty$ and $x \in [0,1]$, then

$$\left(\frac{1-x}{2}\right)^p + \left(\frac{1+x}{2}\right)^p \leq \frac{1+x^p}{2}.$$

This inequality implies that for $a, b \in \mathbb{R}$ that

$$\left(\frac{|a-b|}{2}\right)^p + \left(\frac{|a+b|}{2}\right)^p \leq \frac{|a|^p + |b|^p}{2}.$$

This is obviously true when $a = b = 0$. If one of $a, b \in \mathbb{R}$ is non-zero, say $a \neq 0$, the inequality becomes $\frac{|a|^p}{2^{p-1}} \leq \frac{|a|^p}{2}$, or $1 \leq 2^{p-2}$, which is true since $p \geq 2$. In general, set $x = \frac{|a|}{|b|}$ and rearrange.

Now let $a = f(x)$, $b = g(x)$, and then integrate over E with respect to the measure μ to find:

$$\left\|\frac{f-g}{2}\right\|^p_{L^p(E;\mu)} + \left\|\frac{f+g}{2}\right\|^p_{L^p(E;\mu)} \leq \frac{1}{2}\|f\|^p_{L^p(E;\mu)} + \frac{1}{2}\|g\|^p_{L^p(E;\mu)}.$$

For the lower inequality, proceed essentially as above and prove the following: Let $2 \leq p < \infty$ and $x \in [0,1]$, then

$$2(1+x^p) \leq (1+x)^p + (1-x)^p.$$

This inequality then implies that for $a, b \in \mathbb{R}$ that

$$2^{1-p}\left(|a|^p + |b|^p\right) \leq \left(\frac{|a-b|}{2}\right)^p + \left(\frac{|a+b|}{2}\right)^p.$$

If $a = b = 0$ then this is an obvious inequality. If $a \neq 0$ then it reduces to $2^{1-p}|a|^p \leq 2\frac{|a|^p}{2^p}$, which is obviously true. In the general case, let $x = \frac{|a|}{|b|}$ and rearrange. Now let $a = f(x)$, $b = g(x)$, and then integrate over E with respect to the measure μ to find:

$$2^{1-p}\left(\|f\|^p_{L^p(E;\mu)} + \|g\|^p_{L^p(E;\mu)}\right) \leq \left\|\frac{f-g}{2}\right\|^p_{L^p(E;\mu)} + \left\|\frac{f+g}{2}\right\|^p_{L^p(E;\mu)}.$$

For the upper inequality in this problem, it is possible to give a proof of this using convexity and the Parallelogram Identity.

3. Suppose that $0 < \theta < 1$, $E \subset \mathbb{R}^n$ and $0 < |E| < \infty$. Prove that there is a cube Q such that

$$\theta |Q| < |E \cap Q| .$$

Solution: Fix $0 < \theta < 1$. Then there exists an open set G such that $\theta |G| < |E|$ and $G \supset E$. Since G is open, it is possible to write G as a collection of non-overlapping cubes Q_j, i.e. $G = \bigcup_j Q_j$. We then have

$$\theta \sum_j |Q_j| = \theta |G|$$

Now observe that this gives $\theta \sum_j |Q_j| < \sum_j |E \cap Q_j|$. If not, then we would have that

$$\begin{aligned} \theta |G| &= \theta \sum_j |Q_j| \\ &\geq \sum_j |E \cap Q_j| \\ &= |E| . \end{aligned}$$

Here the first equality follows from the properties of the cube, the middle inequality is our supposition, and the last equality follows since $E = \bigcup_j E \cap Q_j$ with the cubes Q_j disjoint and $G = \bigcup_j Q_j \supset E$. So we have that $\theta |G| \geq |E|$, which is in contradiction to our choice of G from the beginning. So, we now have that

$$\theta \sum_j |Q_j| < \sum_j |E \cap Q_j| .$$

Thus, there must exist at least on integer N such that $\theta |Q_N| < |E \cap Q_N|$. Set $Q = Q_N$ for the desired result that

$$\theta |Q| < |E \cap Q| .$$

4. Let $\{f_n\}, \{g_n\}, f, g \in L^1(E;\mu)$. Suppose that $f_n \to f$ and $g_n \to g$ μ-almost everywhere and that $|f_n| \le g_n$ and $\int_E g_n(x)\,d\mu(x) \to \int_E g(x)\,d\mu(x)$. Show that

$$\int_E f_n(x)\,d\mu(x) \to \int_E f(x)\,d\mu(x).$$

Solution: Since $|f_n| \le g_n$ we have that $g_n \pm f_n$ is non-negative and measurable. Then we have that

$$\begin{aligned}
\int_E g(x)\,d\mu(x) + \int_E f(x)\,d\mu(x) &= \int_E (g(x) + f(x))\,d\mu(x) \\
&= \int_E \varliminf_{n\to\infty} (f_n(x) + g_n(x))\,d\mu(x) \\
&\le \varliminf_{n\to\infty} \int_E (g_n(x) + f_n(x))\,d\mu(x) \\
&= \int_E g(x)\,d\mu(x) + \varliminf_{n\to\infty} \int_E f_n(x)\,d\mu(x).
\end{aligned}$$

Here, the inequality follows from an application of Fatou's Lemma. This computation thus gives that

$$\int_E f(x)\,d\mu(x) \le \varliminf_{n\to\infty} \int_E f_n(x)\,d\mu(x). \tag{1}$$

Now, repeat the computation above, but this time with the function $g - f$ to see,

$$\begin{aligned}
\int_E g(x)\,d\mu(x) - \int_E f(x)\,d\mu(x) &= \int_E (g(x) - f(x))\,d\mu(x) \\
&= \int_E \varliminf_{n\to\infty} (g_n(x) - f_n(x))\,d\mu(x) \\
&\le \varliminf_{n\to\infty} \int_E (g_n(x) - f_n(x))\,d\mu(x) \\
&= \int_E g(x)\,d\mu(x) + \varliminf_{n\to\infty} \int_E (-f_n(x))\,d\mu(x).
\end{aligned}$$

Rearrangement then gives that

$$\int_E f(x)\,d\mu(x) \ge -\varliminf_{n\to\infty} \int_E (-f_n(x))\,d\mu(x) = \varlimsup_{n\to\infty} \int_E f_n(x)\,d\mu(x). \tag{2}$$

Combining (1) and (2) we see that

$$\int_E f(x)\,d\mu(x) \le \varliminf_{n\to\infty} \int_E f_n(x)\,d\mu(x) \le \varlimsup_{n\to\infty} \int_E f_n(x)\,d\mu(x) \le \int_E f(x)\,d\mu(x),$$

and so

$$\lim_{n\to\infty} \int_E f_n(x)\,d\mu(x) = \int_E f(x)\,d\mu(x).$$

5. A sequence of vectors $\{x_n\}$ in a Hilbert space is called a *Riesz sequence* if there exist constants $0 < a \le A < \infty$ such that

$$a \sum_j |a_j|^2 \le \left\| \sum_j a_j x_j \right\|_H^2 \le A \sum_j |a_j|^2 \tag{3}$$

for any collection of numbers $\{a_j\} \in \ell^2$. A Riesz sequence is called a *Riesz basis* if $H = \overline{\operatorname{span}\{x_n\}}$, i.e., the closed linear space of $\{x_n\}$ is the Hilbert space H. Show that $\{x_n\}$ is a Riesz basis for a Hilbert space H if and only if $x_n = Te_n$ where $T : H \to H$ is an isomorphism and $\{e_n\}$ is an orthonormal basis for H.

Solution:
First, suppose that $x_n = Te_n$ where T is an isomorphism and $\{e_n\}$ is an orthonormal basis for H. Then note that

$$\sum_j a_j x_j = \sum_j a_j T e_j = T\left(\sum_j a_j e_j\right).$$

Since T is an isomorphism we have that $c\|h\|_H \le \|Th\|_H \le C\|h\|_H$ (here $C = \|T\|$ and $c = \|T^{-1}\|^{-1}$). Then consider the vector $h = \sum_j a_j e_j$ and note that

$$\|h\|_H^2 = \left\langle \sum_j a_j e_j, \sum_k a_k e_k \right\rangle_H = \sum_j |a_j|^2$$

and since T is an isomorphism we have

$$c^2 \|h\|_H^2 \le \|Th\|_H^2 \le C^2 \|h\|_H^2.$$

But then using the definition of the isomorphism T and the specific choice of vector h we have that

$$c^2 \sum_j |a_j|^2 \le \left\| \sum_j a_j x_j \right\|_H^2 \le C^2 \sum_j |a_j|^2.$$

which is (3) with $a = c^2$ and $A = C^2$. It remains to show that $\overline{\operatorname{span}\{x_n\}} = H$, but this follows from (b) below, so we omit this computation.

Conversely, now suppose that $\{x_n\}$ is a Riesz basis for H. Define a linear operator on finite sums of the $\{e_j\}$ in the following manner

$$T\left(\sum_j a_j e_j\right) := \sum_j a_j x_j.$$

It is easy to see that T is linear on finite sums. Let h be an arbitrary element of H, then we have that

$$h = \sum_{j=1}^{\infty} a_j e_j \quad \text{and} \quad \|h\|_H^2 = \sum_{j=1}^{\infty} |a_j|^2.$$

We now show how to define the operator on any element $h \in H$. Consider $h_N = \sum_{j=1}^N a_j e_j$ and and h_M written similarly with $M > N$. Then we have using the right inequality in (3) that

$$\|Th_M - Th_N\|_H^2 = \left\| T\left(\sum_{j=N+1}^{M} a_j e_j\right)\right\|_H^2 \leq A \sum_{j=N+1}^{M} |a_j|^2 \to 0.$$

So for any $h \in H$ we have the the resulting sequence $\{Th_j\}$ is Cauchy in H, and so we then define

$$Th = T\left(\sum_{j=1}^{\infty} a_j e_j\right) = \lim_{N\to\infty} Th_N = \sum_{j=1}^{\infty} a_j x_j.$$

It is immediate that T as defined in this way is linear on H. Now we show that T is a bounded operator and invertible operator. For the vector $h_N = \sum_{j=1}^N a_j e_j$ using (3) we have that

$$a\sum_{j=1}^{N} |a_j|^2 \leq \|Th_N\|_H^2 \leq A\sum_{j=1}^{N} |a_j|^2$$

which upon passing to the limit gives that

$$a\,\|h\|_H^2 \leq \|Th\|_H^2 \leq A\,\|h\|_H^2\,.$$

This gives that $T : H \to H$ is bounded, with norm less than or equal to $\sqrt{A}$, and that $T^{-1} : \text{Ran } T \to H$ is bounded with norm $\sqrt{a}^{-1}$. Also note that T is injective since if $Th = 0$ then by the left hand inequality above, we have that $\|h\|_H = 0$ and so $h = 0$. It remains to show that Ran $T = H$. To accomplish this note that $Te_n = x_n$, and so $\text{span}\{x_n\} \subset \text{Ran } T$, and so $H = \overline{\text{span}\{x_n\}} \subset \overline{\text{Ran } T}$. It remains to show that Ran T is closed.

Suppose that $y_n \in \text{Ran } T$ and that $y_n \to y$. We can find elements $\{x_n\} \in H$ such that $Tx_n = y_n$, and $\{Tx_n\}$ is Cauchy in H. By the left inequality in (3) we have that

$$a\,\|x_n - x_m\|_H^2 \leq \|Tx_n - Tx_m\|_H \to 0$$

so we have that $\{x_n\}$ is Cauchy in H, and must converge to an element $x \in H$. Since T is continuous we have that Tx_n converges to Tx. But, this implies that $y_n \to Tx$, and $y_n \to y$, or $y = Tx \in \text{Ran } T$. So Ran T is closed and thus $T : H \to H$ is and isomorphism.

6. Suppose that $E \subset \mathbb{R}^n$ with $|E| < \infty$ and let f be a non-negative measurable function on E. Prove that the following are equivalent:

(a) $f \in L^p(E)$;

(b) $\sum_{k=-\infty}^{\infty} 2^{kp} \left|\{x \in E : f(x) > 2^k\}\right| < \infty$.

Solution: Let $\omega(\lambda) = \left|\{x \in E : f(x) > 2^k\}\right|$. Using the distribution function $\omega(\lambda)$ we have that

$$\int_E f(x)^p \, dx = p \int_0^\infty \lambda^{p-1} \omega(\lambda) \, d\lambda. \tag{4}$$

Using this, the result will follow. So first suppose that (a) holds, then we know that the left hand side of (4) is finite. We now proceed to decompose the domain of integration on the right hand side.

$$\begin{aligned} p \int_0^\infty \lambda^{p-1} \omega(\lambda) \, d\lambda &= p \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \lambda^{p-1} \omega(\lambda) \, d\lambda \\ &\geq p \sum_{k=-\infty}^{\infty} \omega(2^{k+1}) \int_{2^k}^{2^{k+1}} \lambda^{p-1} \, d\lambda \\ &= \sum_{k=-\infty}^{\infty} \omega(2^{k+1}) \left(2^{p(k+1)} - 2^{kp}\right) \\ &= (2^p - 1) 2^{-p} \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k). \end{aligned}$$

So we have that

$$\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) < \infty$$

or (b) holds.

Conversely, if (b) holds, then we have that

$$\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) < \infty.$$

We now show that this sum dominates the corresponding integrand. Consider the integral

$$p \int_{2^k}^{2^{k+1}} \lambda^{p-1} \omega(\lambda) \, d\lambda \leq p \omega(2^k) \int_{2^k}^{2^{k+1}} \lambda^{p-1} \, d\lambda = (2^p - 1) 2^{kp} \omega(2^k).$$

Summing this expression over $k \in \mathbb{Z}$ gives

$$p \int_0^\infty \lambda^{p-1} \omega(\lambda) \, d\lambda \leq (2^p - 1) \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k).$$

So we have

$$p \int_0^\infty \lambda^{p-1} \omega(\lambda) \, d\lambda < \infty$$

this gives that $f \in L^p(E)$.

7. We say that a function $f\colon [a,b] \to \mathbb{R}$ is Lipschitz if there exists a constant $K \geq 0$ such that $|f(x) - f(y)| \leq K\,|x-y|$ for all $x,\, y \in [a,b]$.

(a) Prove that f is Lipschitz if and only if f is absolutely continuous on $[a,b]$ and $f' \in L^\infty[a,b]$.

(b) Let $\mathrm{Lip}[a,b]$ denote the space of Lipschitz functions on $[a,b]$. Prove that $\mathrm{Lip}[a,b]$ is a meager subset (also sometimes called a set of Baire first category) of $C[a,b]$.

Solution:

Solution to part (a). $\Rightarrow$. Suppose that f is Lipschitz. Fix $\varepsilon > 0$, and let $\delta = \varepsilon/K$. Let $\{[a_j,b_j]\}_j$ be any countable collection of nonoverlapping subintervals of $[a,b]$ such that

$$\sum_{j=1}^{m} (b_j - a_j) < \delta.$$

Then

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| \leq K \sum_{j=1}^{m} (b_j - a_j) < K\delta = \varepsilon.$$

Hence f is absolutely continuous on $[a,b]$. As a consequence, f' exists a.e. and is integrable. Further, for each x where the derivative exists,

$$|f'(x)| = \lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq \lim_{h\to 0} \left| \frac{K\,(x+h-x)}{h} \right| = K.$$

Hence $|f'| \leq K$ a.e., so $f' \in L^\infty[a,b]$.

$\Leftarrow$. Assume that f is absolutely continuous and there is a constant M such that $|f'| \leq M$ a.e. Absolutely continuous functions satisfy the Fundamental Theorem of Calculus, so if $a \leq x < y \leq b$ then

$$|f(x) - f(y)| = \left| \int_x^y f'(t)\,dt \right| \leq \int_x^y |f'(t)|\,dt \leq \int_x^y M\,dt = M\,(y-x).$$

Hence f is Lipschitz.

Solution to part (b). Let F_n be the set of functions that are Lipschitz with Lipschitz constant at most n:

$$F_n = \{f \in C[a,b] : |f(x) - f(y)| \leq n\,|x-y|,\ x,y \in [a,b]\}.$$

Suppose that $f_k \in F_n$ and $f_k \to f$ in $C[a,b]$. Since uniform convergence implies pointwise convergence, we have

$$|f(x) - f(y)| = \lim_{k\to\infty} |f_k(x) - f_k(y)| \leq n\,|x-y|.$$

Hence $f \in F_n$, so F_n is closed.

However, F_n has no interior. If $f \in F_n$ and $\varepsilon > 0$ is given, we can create a function h that equals f everywhere except on a small interval, and on that interval the graph of h is a triangle of height $\frac{\varepsilon}{2}$ with the slope of at least one side greater than n. Then $\|f-h\|_\infty < \varepsilon$ but $h \notin F_n$.

Since $\mathrm{Lip}[a,b] = \cup F_n$, it follows that $\mathrm{Lip}[a,b]$ is a meager subset of $C[a,b]$.

8. Let X, Y be Banach spaces, and let $A\colon X \to Y$ be a bounded linear operator. Prove the existence of the adjoint of A. That is, prove that there is a unique operator $A^*\colon Y^* \to X^*$ that satisfies

$$(A^*\mu)(f) = \mu(Af) \qquad \text{for all } f \in X,\ \mu \in Y^*,$$

and furthermore prove that A^* is bounded, linear, and has operator norm $\|A^*\| = \|A\|$.

Solution: For this solution, we will write $\langle f, \mu\rangle$ instead of $\mu(f)$ to denote the action of a functional μ on a vector f. For simplicity, we shall assume that scalars in this problem are real, but only minor changes are needed if we allow complex scalars.

Construction of $A^*\mu$. With $\mu \in Y^*$ fixed, define $A^*\mu\colon X \to \mathbb{R}$ by

$$\langle f, A^*\mu\rangle = \langle Af, \mu\rangle, \quad f \in X.$$

The operator $A^*\mu$ is a functional, and it is a linear function of f because A and μ are both linear. Also,

$$|\langle f, A^*\mu\rangle| = |\langle Af, \mu\rangle| \le \|Af\|\ \|\mu\| \le \|A\|\ \|f\|\ \|\mu\|,$$

so

$$\|A^*\mu\| = \sup_{\|f\|=1} |\langle f, A^*\mu\rangle| \le \|A\|\ \|\mu\|.$$

Hence $A^*\mu$ is a bounded linear functional on X, and therefore $A^*\mu \in X^*$.

Boundedness and Linearity of A^*. Given any $\mu \in Y^*$ we have defined a functional $A^*\mu \in X^*$. Now consider the mapping $A^*\colon Y^* \to X^*$ that takes μ to $A^*\mu$. If μ, $\nu \in Y^*$ and a, $b \in \mathbb{R}$, then $A^*(a\mu + b\nu)$ is the functional defined by

$$\begin{aligned}
\langle f, A^*(a\mu + b\nu)\rangle &= \langle Af, a\mu + b\nu\rangle \\
&= a\,\langle Af, \mu\rangle + b\,\langle Af, \nu\rangle \\
&= a\,\langle f, A^*\mu\rangle + b\,\langle f, A^*\nu\rangle \\
&= \langle f, aA^*\mu + bA^*\nu\rangle.
\end{aligned}$$

Therefore $A^*(a\mu + b\nu) = aA^*\mu + bA^*\nu$, so A^* is a linear mapping. Further, we showed above that $\|A^*\mu\| \le \|A\|\ \|\mu\|$, so A^* is bounded and $\|A^*\| \le \|A\|$.

Norm of A^*. Choose any vector $f \in X$ with $\|f\| = 1$. By Hahn–Banach,

$$\|Af\| = \sup_{\|\mu\|=1} |\langle Af, \mu\rangle|,$$

and this supremum is achieved, say by the unit functional μ. Therefore

$$\begin{aligned}
\|Af\| &= |\langle Af, \mu\rangle| \\
&= |\langle f, A^*\mu\rangle| \\
&\le \|f\|\ \|A^*\mu\| \\
&\le \|f\|\ \|A^*\|\ \|\mu\| \\
&= \|f\|\ \|A^*\|.
\end{aligned}$$

Since this is true for every unit vector $f \in X$, we conclude that $\|A\| \leq \|A^*\|$. Since we proved above that $\|A^*\| \leq \|A\|$, it follows that the operator norms of A and A^* are equal.

Uniqueness of the Adjoint. Suppose that $B \colon Y^* \to X^*$ also satisfies

$$\forall\, f \in X, \quad \forall\, \mu \in Y^*, \quad \langle Af, \mu \rangle = \langle f, B\mu \rangle.$$

Note that we are not assuming that B is linear or bounded. With $\mu \in Y^*$ fixed, we have

$$\langle f, A^*\mu - B\mu \rangle = 0, \qquad f \in X.$$

Hence $A^*\mu - B\mu$ is the zero operator. Thus $A^*\mu = B\mu$ for every $\mu \in Y^*$, so $B = A^*$.