Analysis Comprehensive Exam Questions
Spring 2014

NOTE: Throughout this exam, the Lebesgue exterior measure of a set $E \subseteq \mathbb{R}^d$ will denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is denoted by $|E|$.

1. Let K be a compact subset in $\mathbb{R}^d$, and let $f(x) = \text{dist}(x, K)$. Let $g(x) = \max\{1 - f(x), 0\}$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g(x)^n \, dx = |K|.$$

Solution: Note that $f(x) = 0$ for all $x \in K$ and $f(x) > 0$ for all $x \notin K$. Let K_1 be the set of points that are a distance of at most 1 from the set K, i.e., $K_1 = \{x : f(x) \le 1\}$. Then we have $K \subseteq K_1$ and

$$g(x) = (1 - f)\, \chi_{K_1}.$$

Since g is identically zero outside the set K_1, we have $g^n \to 0$ on $\mathbb{R}^d \setminus K_1$. On the set $K_1 \setminus K$ we have that $0 \le 1 - f(x) < 1$, so $g^n \to 0$ on this set. On the set K we have $g^n = 1$. Hence $g^n \to \chi_K$. Finally, observe that $g^n \le \chi_{K_1} \in L^1(\mathbb{R}^d)$. So, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g(x)^n \, dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} g(x)^n \, dx = \int_{\mathbb{R}^d} \chi_K(x) \, dx = |K|.$$

2. (a) Let A be any subset of $\mathbb{R}^d$. Prove that there exists a measurable set $H \supseteq A$ that satisfies

$$|A \cap E|_e = |H \cap E| \quad \text{for every measurable set } E \subseteq \mathbb{R}^d.$$

Solution: *Case 1*: $|A|_e < \infty$. For each $n > 0$, there exists an open set $U_n \supseteq A$ such that $|A|_e \le |U_n| < |A|_e + \frac{1}{n}$. Therefore $H = \cap U_n$ is a G_δ-set that contains A and satisfies $|A|_e = |H|$.

Now let E be any measurable subset of $\mathbb{R}^d$. Applying the Carathéodory Criterion and monotonicity, we see that

$$|H \cap E| + |H \backslash E| = |H| = |A|_e = |A \cap E|_e + |A \backslash E|_e \le |H \cap E| + |H \backslash E|. \tag{1}$$

Therefore equality holds in (1). However, monotonicity implies $|A \cap E|_e \le |H \cap E|$ and $|A \backslash E|_e \le |H \backslash E|$, so since all of the quantities involved are finite, the only way that equation (1) can hold is if

$$|A \cap E|_e = |H \cap E| \qquad \text{and} \qquad |A \backslash E|_e = |H \backslash E|.$$

Case 2: Arbitrary sets. For each $k \in \mathbb{N}$, set

$$A_k = A \cap [-k, k]^d.$$

Each set A_k has finite measure, and $A = \cup A_k$. By Case 1, for each k there is a G_δ-set $H_k \supseteq A_k$ such that

$$|A_k \cap E|_e = |H_k \cap E| \quad \text{for every measurable } E.$$

The sets

$$G_j = \bigcap_{k=j}^{\infty} H_k, \qquad j \in \mathbb{N},$$

are nested, and their union

$$H = \bigcup_{j=1}^{\infty} G_j$$

is measurable. If E is any measurable subset of $\mathbb{R}^d$, then

$$\begin{aligned}
|H \cap E| = \left|\bigcup_{j=1}^{\infty} (G_j \cap E)\right| &= \lim_{j\to\infty} |G_j \cap E| && \text{(continuity from below)} \\
&\le \limsup_{j\to\infty} |H_j \cap E| && \text{(since } G_j \subseteq H_j\text{)} \\
&= \limsup_{j\to\infty} |A_j \cap E|_e && \text{(by definition of } H_j\text{)} \\
&\le |A \cap E|_e && \text{(since } A_j \subseteq A\text{)} \\
&\le |H \cap E| && \text{(since } A \subseteq H\text{)}.
\end{aligned}$$

3. Let $\{f_n\}$ be a sequence in $L^p(\Omega)$ $(1 \le p \le \infty)$ and assume that $f \in L^p(\Omega)$ is such that $\|f_n - f\|_p \to 0$. Show that there exists a subsequence $\{f_{n_k}\}$ and a function $h \in L^p(\Omega)$ such that

(i) $f_{n_k}(x) \to f(x)$ a.e. on Ω.

(ii) for every k we have $|f_{n_k}(x)| \le h(x)$ a.e. on Ω.

Solution: The conclusion is obvious when $p = \infty$. So we assume $1 \le p < \infty$. Since $\{f_n\}$ is a Cauchy sequence in $L^p(\Omega)$, we can extract a subsequence $\{f_{n_k}\}$ such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\|_p \le \frac{1}{2^k}, \quad \forall k \ge 1.$$

To simplify notations, we denote f_{n_k} by f_k. Let

$$g_n(x) = \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)|,$$

and observe that $\|g_n\|_p \le 1$. By the Monotone Convergence Theorem, $g_n(x)$ tends to a finite limit $g(x)$ a.e. on Ω, and $g \in L^p(\Omega)$. On the other hand, for any $m \ge n \ge 2$, we have

$$\begin{aligned}|f_m(x) - f_n(x)| &\le |f_m(x) - f_{m-1}(x)| + \cdots + |f_{n+1}(x) - f_n(x)| \\ &\le g(x) - g_{n-1}(x).\end{aligned}$$

Thus, a.e. on Ω, $\{f_n(x)\}$ is Cauchy and therefore converges to a finite limit $f^*(x)$. We have a.e. on Ω,

$$|f^*(x) - f_n(x)| \le g(x), \text{ for } n \ge 2.$$

In particular, $f^* \in L^p(\Omega)$. Since

$$|f_n(x)| \le g(x) + |f^*(x)| = h(x), \text{ with } h \in L^p(\Omega),$$

the Dominated Convergence Theorem implies that $f_k \to f^*$ in $L^p(\Omega)$, and therefore $f^* = f$ a.e. This finishes the proof.

4. Suppose that $f \in L^1(\mathbb{R})$ is such that $f' \in L^1(\mathbb{R})$ and f is absolutely continuous on every finite interval $[a,b]$. Show that $\lim_{x\to\infty} f(x) = 0$.

Solution: Suppose that $f(x)$ does not converge to zero as $x \to \infty$. This does not say that f must converge to some other value as $x \to \infty$, but it does tell us that there exists some $\varepsilon > 0$ such that:

$$\forall\, R > 0, \quad \exists\, x > R \text{ such that } |f(x)| > 2\varepsilon. \tag{2}$$

Since f' is integrable, there exists a $\delta > 0$ such that for any measurable set $A \subseteq \mathbb{R}$,

$$|A| < \delta \implies \int_A |f'| < \varepsilon.$$

Fix any points $x < y$ such that $y - x < \delta$. Since f is absolutely continuous on $[x, y]$,

$$|f(y) - f(x)| = \left| \int_x^y f' \right| \le \int_x^y |f'| < \varepsilon.$$

(Note that this shows that f is uniformly continuous on $\mathbb{R}$.)

By equation (2), there exists some point $x_1 > 1$ such that $|f(x_1)| > 2\varepsilon$. Hence if $x \in (x_1 - \delta, x_1 + \delta)$ then

$$2\varepsilon < |f(x_1)| \le |f(x_1) - f(x)| + |f(x)| < \varepsilon + |f(x)|.$$

Thus $|f(x)| > \varepsilon$ on the interval $(x_1 - \delta, x_1 + \delta)$.

Now we repeat this argument. There exists some $x_2 > x_1 + \delta$ such that $|f(x_2)| > 2\varepsilon$. As before we find that $|f(x)| > \varepsilon$ on the interval $(x_2 - \delta, x_2 + \delta)$. Continuing in this way, f is bounded below by ε on infinitely many disjoint intervals of length 2δ, which implies that f is not integrable.

5. Let E be a measurable subset of $\mathbb{R}^d$ such that $|E| < \infty$, and let f be any nonnegative, bounded function on E. Suppose that

$$\sup\left\{\int_E \phi \,:\, 0 \le \phi \le f,\ \phi \text{ simple}\right\} = \inf\left\{\int_E \psi \,:\, f \le \psi,\ \psi \text{ simple}\right\}.$$

Prove that f is measurable.

Solution: Let

$$I = \sup\left\{\int_E \phi \,:\, 0 \le \phi \le f,\ \phi \text{ simple}\right\} = \inf\left\{\int_E \psi \,:\, f \le \psi,\ \psi \text{ simple}\right\}.$$

Since f is bounded, there is a constant M such that $f \le M$. Therefore, if ϕ is a simple function and $\phi \le f$, then $\phi \le M$. Consequently

$$\int_E \phi \le \int_E M = M\,|E|,$$

and therefore $I \le M\,|E| < \infty$.

Now, by the definition of sup and inf, there exist simple functions

$$0 \le \phi_n \le f \le \psi_n$$

such that

$$\lim_{n\to\infty} \int_E \phi_n = I = \lim_{n\to\infty} \int_E \psi_n.$$

Set

$$\phi = \sup_n \phi_n \qquad \text{and} \qquad \psi = \inf_n \psi_n.$$

Then ϕ and ψ are each measurable and nonnegative, and for every $n \in \mathbb{N}$ we have

$$0 \le \phi_n \le \phi \le f \le \psi \le \psi_n.$$

Although we do not know whether f is measurable, both ϕ and ψ are measurable, so

$$\int_E \phi_n \le \int_E \phi \le \int_E \psi \le \int_E \psi_n$$

for every $n \in \mathbb{N}$. Consequently,

$$I = \lim_{n\to\infty} \int_E \phi_n \le \int_E \phi \le \int_E \psi \le \lim_{n\to\infty} \int_E \phi_n = I.$$

As ϕ is integrable, we have

$$\int_E (\psi - \phi) = \int_E \psi - \int_E \phi = 0.$$

But $\psi - \phi$ is nonnegative, so this implies that $\psi - \phi = 0$ a.e. Finally, $\phi \le f \le \psi$, so we conclude that $\phi = f = \psi$ a.e. Therefore f is measurable.

6. Assume that $\{e_n\}$ is an orthonormal basis for a Hilbert space H. Let $\{a_n\}$ be a bounded sequence in $\mathbf{R}$, and set

$$u_n = \frac{1}{n} \sum_{i=1}^{n} a_i e_i.$$

Show that:

(i) $|u_n| \to 0$,

(ii) $\sqrt{n} u_n \rightharpoonup 0$ weakly in H.

Solution: (i) Assume $|a_n| \le M$, $\forall n$. Then

$$\left|\sqrt{n}\, u_n\right|^2 = \frac{1}{n} \sum_{i=1}^{n} |a_i|^2 \le M^2.$$

Thus

$$|u_n| \le \frac{M}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty.$$

(ii) Take any fixed e_j, then

$$|\langle \sqrt{n}\, u_n, e_j \rangle| = \frac{1}{\sqrt{n}} |a_j| \le \frac{M}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty.$$

This extends to all vectors in H by a density argument.

To be more explicit, if y is any finite linear combination of the e_j, then, by linearity,

$$\langle \sqrt{n}\, u_n, y \rangle \to 0.$$

If z is an arbitrary element of H and we fix $\varepsilon > 0$, then there exists some $y \in \operatorname{span}\{e_j\}$ such that $|y - z| < \varepsilon$. There is some $N > 0$ such that for all $n > N$ we have

$$|\langle \sqrt{n}\, u_n, y \rangle| < \varepsilon.$$

Therefore, for all $n > N$,

$$|\langle \sqrt{n}\, u_n, z \rangle| \le |\langle \sqrt{n}\, u_n, z - y \rangle| + |\langle \sqrt{n}\, u_n, y \rangle| \le |\sqrt{n}\, u_n|\, |z - y| + \varepsilon \le M\,\varepsilon + \varepsilon.$$

This shows that $\langle \sqrt{n}\, u_n, z \rangle \to 0$.

7. Let ν be a bounded signed Borel measure on $\mathbb{R}$, and assume $f \in L^1(\mathbb{R})$ (i.e., f is integrable with respect to Lebesgue measure). Prove that

$$g(x) = \int_{-\infty}^{\infty} f(x-y)\, d\nu(y)$$

is defined at almost every x and

$$\|g\|_1 \leq \|f\|_1 \, |\nu|(\mathbb{R}),$$

where $|\nu|$ is the total variation of ν. Show further that if f is uniformly continuous on $\mathbb{R}$, then so is g.

Solution: Given $x \in \mathbb{R}$,

$$\iint |f(x-y)|\, dx\, d|\nu|(y) = \int \left(\int |f(x)|\, dx \right) d|\nu|(y) = \|f\|_1 \int d|\nu(y)| = \|f\|_1 \, |\nu|(\mathbb{R}).$$

Therefore the integral defining $g(x)$ exists a.e. Furthermore, Fubini's Theorem implies that g is Lebesgue measurable. Using Fubini's Theorem again, we compute that

$$\begin{aligned} \|g\|_1 = \int \left| \int f(x-y)\, d\nu(y) \right| dx &\leq \iint |f(x-y)|\, d|\nu|(y)\, dx \\ &\leq \iint |f(x-y)|\, dx\, d|\nu|(y) \\ &\leq \|f\|_1 \, |\nu|(\mathbb{R}). \end{aligned}$$

Now suppose that f is uniformly continuous. Let $T_a f(x) = f(x+a)$, and let $\| \cdot \|_\infty$ denote the uniform norm. Then

$$\begin{aligned} |g(x+a) - g(x)| &= \left| \int \big(f(x+a-y) - f(x-y) \big)\, d\nu(y) \right| \\ &\leq \int |f(x+a-y) - f(x-y)|\, d|\nu(y)| \\ &\leq \int \|T_a f - f\|_\infty \, d|\nu|(y) \\ &= \|T_a f - f\|_\infty \, |\nu|(\mathbb{R}). \end{aligned}$$

Therefore, since f is uniformly continuous,

$$\|T_a g - g\|_\infty \leq \|T_a f - f\|_\infty \, |\nu|(\mathbb{R}) \to 0 \quad \text{as } a \to 0.$$

Hence g is uniformly continuous.

8. Prove that

$$Vf(x) = \int_0^x f(y)\,dy$$

is a continuous, injective mapping of $L^p[0,1]$ into itself for each $1 \le p \le \infty$.

Solution: The mapping V is linear, and we will prove that it maps $L^p[0,1]$ boundedly into itself for $1 < p < \infty$ (the cases $p = 1$ and $p = \infty$ are similar). Fix $f \in L^p[0,1]$, and note that f is integrable since $[0,1]$ has finite measure. Letting p' denote the dual index to p, we compute that

$$\begin{aligned}
\|Vf\|_p^p = \int_0^1 |Vf(x)|^p\,dx &= \int_0^1 \left|\int_0^x f(y)\,dy\right|^p dx \\
&\le \int_0^1 \left(\int_0^x |f(y)|^p\,dy\right)\left(\int_0^x 1^{p'}\,dy\right)^{p/p'} dx \\
&\le \int_0^1 \left(\int_0^1 |f(y)|^p\,dy\right) \cdot 1\,dx \\
&= \left(\int_0^1 |f(y)|^p\,dy\right) \\
&= \|f\|_p^p.
\end{aligned}$$

Therefore V is bounded (and hence continuous).

Suppose that $Vf = 0$ a.e. Since Vf is continuous, this implies that Vf is identically zero. However, Vf is absolutely continuous, so the Fundamental Theorem of Calculus implies that

$$f = (Vf)' = 0 \text{ a.e.}$$

Therefore V is injective.