Topology Comprehensive Exam — Spring 2015
January 14, 2015

Student Number:

Instructions: Complete five of the eight problems below, and **circle** their numbers exactly in the box below – the uncircled problems will **not** be graded.

1	2	3	4	5	6	7	8

Please note that a complete solution of a problem is preferable to partial progress on several problems.
Please write **only on the front side** of the solution pages. Work on the back of the page might not be graded.

1. Let $g : \mathbb{P}^n \to \mathbb{P}^m$ be a map from real projective space of dimension n to that of dimension m and denote by p_n,p_m the 2-fold covering maps $S^n \to \mathbb{P}^n$ and $S^m \to \mathbb{P}^m$ respectively. Assume $n > 1$,$m > 0$.

   (a) Prove that there is a map $f : S^n \to S^m$ such that $p_m f = g p_n$. Further show that either $f(-x) = f(x)$ for all x ("f is even") or $f(-x) = -f(x)$ for all x ("f is odd").

   (b) Prove that the function f in (a) is even precisely when the induced map on fundamental groups $g_* : \pi_1(\mathbb{P}^n) \to \pi_1(\mathbb{P}^m)$ is trivial and is an odd function precisely when g_* is an isomorphism.

   (c) Use (b) and the fact, which you may assume, that an odd map has odd degree, to prove that when $n > m$, then g_* is always the trivial homomorphism.

2. Let T be the torus $S^1 \times S^1$ and $f : S^1 \to T : \theta \mapsto (\theta, (1, 0))$ for the point $(1, 0) \in S^1$. Let X be the space obtained by attaching a 2-cell D^2 to T with the map f.

   1. Let S^2 be the 2-sphere. Show there exists maps $\phi : S^2 \to X$ and $\psi : X \to S^2$ both of which are not homotopic to a constant map. (Hint: consider their composition and degree theory.)
   2. Show that any map S^2 to T is homotopic to a constant map.

3. Let Σ be a smooth submanifold of $\mathbb{R}^n$ of co-dimension bigger than 2. Show that $\mathbb{R}^n - \Sigma$ is connected and simply connected (recall this means that any continuous map of S^1 into the space is homotopic to a constant loop).

4. Let (X, x_0) and (Y, y_0) be path-connected, locally path-connected, and semi-locally simply connected, pointed topological spaces. Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. Show that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective if for any path connected pointed covering map $p : (E, e_0) \to (Y, y_0)$, the pull-back $(E \times_Y X, e_0 \times x_0) \to (X, x_0)$ is path connected, where $E \times_Y X = \{(e, x) \in E \times X : f(x) = p(e)\}$.

5. Let M be an $n-1$ dimensional compact submanifold of $\mathbb{R}^n$ not containing the origin. Show that for almost all directions v in the unit sphere, the ray $\{vt : t \in \mathbb{R}_{\geq 0}\}$ intersects M in only finitely many points.

6. Let ω be a 1-form on a 3-dimensional manifold M. Suppose that ω is not zero at any point so for each $x \in M$ the kernel ξ_x of $\omega(x)$ is a plane in $T_x M$. We say that ξ is integrable if for any two vector fields v and w with values in ξ (that is v and w are sections of ξ) we have that the Lie bracket $[v, w]$ is also a section of ξ. For this problem assume that ω is integrable.

   1. Show that $\omega \wedge d\omega = 0$.
   2. Show there exists a 1-form α such that $d\omega = \omega \wedge \alpha$. (Hint: prove this locally and then use a partition of unity.)
   3. Show that $\omega \wedge d\alpha = 0$.
   4. If β is another 1-form such that $d\omega = \omega \wedge \beta$ then there is a function f such that $\beta = \alpha + f\omega$ and $\alpha \wedge d\alpha = \beta \wedge d\beta$.

7. Consider the form $\alpha = (x^2+y^2+z^2)^{-3/2}(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$ on $\mathbb{R}^3 - \{(0,0,0)\}$ with Euclidean coordinates (x, y, z). Let $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$.

   1. Show that α is closed on $\mathbb{R}^3 - \{(0, 0, 0)\}$.
   2. Compute $\int_{S^2} \alpha$.
   3. Show α is closed but not exact on S^2.
   4. Let Σ be any compact surface embedded in $\mathbb{R}^3 - \{(0, 0, 0)\}$. What are all the possible values of $\int_\Sigma \alpha$. Prove your answer. (You may use the fact that such a surface bounds a compact region K in $\mathbb{R}^3$.)

8. Define a homomorphism $\phi : \mathbb{Z}/2 * \mathbb{Z}/2 \to \Sigma_4$ from the free product of two copies of $\mathbb{Z}/2$, the first with non-zero element called a and the second with the non-zero element called b, to the permutation group on the set $\{1, 2, 3, 4\}$ by

$$\phi(a) = (2, 3)$$

$$\phi(b) = (1, 2)(3, 4).$$

   Let H be the subgroup of $\mathbb{Z}/2 * \mathbb{Z}/2$ whose image under ϕ stabilizes 1, i.e. $H = \{x : \phi(x)1 = 1\}$. Use covering space theory to find the index of H in its normalizer.