Algebra Comprehensive Exam
Fall 2018

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1.
   1. Let G be a finite group of order n, and let $\rho : G \to \mathrm{Sym}(G)$ be the homomorphism that arises from G acting on itself by left translation. Let g in G have order m. Prove that the sign of $\rho(g)$ is $(-1)^{n+n/m}$
   2. Let G be a group of order $2k$ with k odd. Prove that G is a semi-direct product $N \rtimes \mathbb{Z}/2\mathbb{Z}$ where N has order k. (Hint: Using part (1) construct a nontrivial homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$.)

2.
   1. Explicitly exhibit an element σ of $G = \mathrm{GL}_2\, \mathbb{Z}/7\mathbb{Z}$ of order 8.
   2. Describe (with proof) the structure of a 2-Sylow subgroup of G.

   (Hint: Think about the multiplicative subgroup of the field with 49 elements, and the action of the nontrivial automorphism of this field.)

3. Let $F \subseteq E$ be an algebraic extension. Show that $F \subseteq E$ is primitive (i.e., $E = F[\theta]$) if and only if the set of intermediate fields L, $F \subseteq L \subseteq E$, is finite.

4.
   1. Let ζ be a primitive 7th root of unity in $\mathbf{C}$ and let $\beta = \zeta + \zeta^2 + \zeta^4$. Show that $[\mathbf{Q}(\beta) : \mathbf{Q}] = 2$, and that $\sqrt{-7} \in \mathbf{Q}(\beta)$. (Hint: find a linear relation between 1,β, and β^2.)
   2. Let E be the splitting field of the polynomial $f(x) = x^{14} + 7$ over $\mathbf{Q}$ and let α be a root of $f(x)$ in $\mathbf{C}$. Show that $E = \mathbf{Q}[\zeta, \alpha]$ and find the degrees $[E : \mathbf{Q}]$, $[E : \mathbf{Q}(\zeta)]$, and $[E : \mathbf{Q}(\alpha)]$.
   3. Write down elements σ and τ of orders 6 and 7, respectively, in $\mathrm{Gal}(E/\mathbf{Q})$ by explicitly giving the values $\sigma(\zeta)$, $\sigma(\alpha)$, and $\tau(\zeta)$, $\tau(\alpha)$.

5. Prove that every matrix A over an algebraically closed field k can be uniquely written as a sum $A = A_s + A_n$ such that

   - A_s is diagonalizable,
   - A_n is nilpotent (i.e. has all eigenvalues 0), and,
   - $A_n A_s = A_s A_n$.

   (Hint: First prove that for any finite set $a_1, a_2, \ldots, a_n$ of pairwise distinct elements of k and corresponding set of positive integers $m_1, m_2, \ldots, m_n$, there exists a polynomial p such that $p(T) \equiv a_i \mod (T - a_i)^{m_i}$ and set $A_s = p(A)$ and $A_n = A - p(A)$.)

6. Let p be the smallest prime dividing the order of a finite group G. If $P \in \mathrm{Syl}_p(G)$ and P is cyclic prove that $N_G(P) = C_G(P)$.

7. Prove that $f(x) = x^4 + 1$ is reducible modulo every prime p but irreducible in $\mathbb{Q}[x]$.

8. Using the fact that there are infinitely many primes congruent to $1 \mod m$ for any integer m, prove that every finite abelian group appears as the Galois group of some finite Galois extension of $\mathbb{Q}$.