Differential Equations Comprehensive Exam
August 31, 2016

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Prove that the system
$$\begin{cases} \dot{x} = x - y - x^3, \\ \dot{y} = x + y - y^3 \end{cases}$$
has at least one limit cycle.

2. Let $f(x,t)$ be a continuous function on $\mathbb{R} \times \mathbb{R}$. Assume that the initial value problem
$$\begin{cases} \dot{x} = f(x,t) \\ x(0) = 0 \end{cases} \tag{1}$$
admits two solutions $x_1(t)$ and $x_2(t)$ with $x_1(t) \leq x_2(t)$ for $t \in [0, \bar{t}]$. Show that for every $y \in [x_1(\bar{t}), x_2(\bar{t})]$ there is a solution $\bar{x}$ of (1) defined in $[0, \bar{t}]$ with $\bar{x}(\bar{t}) = y$ and $\bar{x}(t) \in [x_1(t), x_2(t)]$.

3. Consider the Initial Value Problem
$$\begin{cases} \dot{x} = Ax + g(x) \\ x(0) = x_0 \end{cases}$$
where $x \in \mathbb{R}^n$, A is a $n \times n$ negative definite matrix and g is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with
$$\lim_{x \to 0} \frac{g(x)}{\|x\|} = 0.$$
Let $\lambda = \max_i \lambda_i$ where λ_i are the eigenvalue of A. Show that for every ϵ there exists δ such that if $\|x_0\| \leq \delta$ then $\|x(t)\| \leq e^{(\lambda+\epsilon)t}\|x_0\|$ for every $t > 0$.

4. Let $f_i(x)$, $i = 1, 2$, be two smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and let $\Phi_i(t,x)$ be the flow generated by $f_i(x)$, that is, $\Phi_i(t,x)$ solves
$$\begin{cases} \dot{\Phi}_i(t,x) = f_i\left(\Phi_i(t,x)\right), \\ \Phi_i(0,x) = x. \end{cases}$$
Show that
$$\Phi_2(t,x) - \Phi_1(t,x) = \int_0^t \mathcal{D}_2\left(t-s, \Phi_1(s,x)\right)\left(f_2 - f_1\right)\left(\Phi_1(s,x)\right) ds$$
where
$$\mathcal{D}_2(t,x) = \frac{\partial \Phi_2(t,x)}{\partial x}.$$

5. Find the solution of the differential equation

$$x_2\partial_{x_1}u(x_1,x_2) + x_1\partial_{x_2}u(x_1,x_2) = (x_1^2 - x_2^2)u(x_1,x_2) \tag{2}$$

in the region $x_1 > 0$, $|x_2| < x_1$ that satisfies

$$u(x_1, 0) = h(x_1).$$

6. Let $u(x,t)$ be a classical solution to the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n, \\ u_t(x,0) = g(x) & \text{in } \mathbb{R}^n. \end{cases}$$

If both f, g vanish for $|x| < R$, prove that $u(0,t) = 0$ for all $t \in [0, R)$.

7. Solve the Burger's equation

$$u_t + uu_x = 0$$

with initial data

$$u(x,0) = \begin{cases} 1 + x & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

8. Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D, and let $a(x)$ be a continuous function on $\bar{D}$. Assume that u is a classical solution to

$$\begin{aligned} &u_t = \Delta u + a(x)u \quad \text{in } D \times (0,\infty) \\ &u = 0 \quad \text{on } \partial D \times (0,\infty), \end{aligned}$$

with non-negative initial condition. Prove that u remains nonnegative for all $t > 0$.