Algebra Comprehensive Exam
January 13, 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Find representatives for the (distinct) conjugacy classes of matrices with characteristic polynomial
$$f(\lambda) = (\lambda^2 + 1)^2$$
in (1) $\mathrm{GL}_4(\mathbf{Q})$ and (2) $\mathrm{GL}_4(\mathbf{C})$.

2. Consider the polynomial
$$f(x) = \frac{x^{23} - 1}{x - 1} = \sum_{n=0}^{22} x^n.$$
Determine the number of irreducible factors of $f(x)$ over (1) $\mathbf{Q}$ (2) $\mathbf{F}_2$ (3) $\mathbf{F}_{2048}$.

3. Which of the following are the class equations for a group G of order 10?
$$1+1+1+2+5 \qquad 1+2+2+5 \qquad 1+2+3+4 \qquad 1+1+2+2+2+2$$
Give an example of a group with each possible class equation, and explain why the rest are impossible.

4. Prove or disprove: the following rings are isomorphic:
$$\mathbf{F}_5[x]/\langle x^4 + x^2 + 1\rangle \qquad \mathbf{F}_5[x]/\langle x^4 - x^3 + x^2 - 1\rangle.$$

5. Give an example of a module M over $\mathbf{Z}[x]$ which is torsion free (for all $f \in \mathbf{Z}[x]$ and $m \in M$, $f \cdot m = 0$ implies $f = 0$ or $m = 0$), but not free.

6. Find the Galois group of the splitting field of the polynomial $f(x) = x^3 - x + 1$:
   1. over $\mathbf{F}_2$
   2. over $\mathbf{R}$
   3. over $\mathbf{Q}$

7. Let R be a commutative ring with 1, and let M be a principal maximal ideal.
   1. Show that there is no ideal I such that $M^2 \subsetneq I \subsetneq M$.
   2. Give an example of a ring R and a maximal ideal M to show this statement is false if M is not assumed principal.

8. Suppose that B is a commutative ring, and that
$$f = \sum_{i=0}^{n} b_i x^i \in B[x]$$
is a polynomial with coefficients in B. Prove that f is nilpotent if and only if all $b_i \in B$ are nilpotent.