Algebra Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let H be the subgroup of S_6 generated by (16425) and $(16)(25)(34)$. Let H act on S_6 by conjugation. Show that the set

$$\Sigma = \{(12)(35)(46), (13)(24)(56), (14)(25)(36), (15)(26)(34), (16)(23)(45)\}$$

is invariant under H, thereby defining a homomorphism $\phi : H \to S_5$. Show that ϕ is an isomorphism.

2. Show that every finite group is isomorphic to a subgroup of a simple group.

3. Let R be a commutative ring with 1. Suppose an ideal I in R is such that $xy \in I$ implies that either $x \in I$ or $y^n \in I$. Let

$$\sqrt{I} = \{r \in R : r^n \quad \text{for} \quad \text{some} \quad n \in \mathbb{Z}_{>}0\}$$

Show that $\sqrt{I}$ is the smallest prime ideal containing I. (Here "smallest" means that any other prime ideal containing I, contains $\sqrt{I}$. Hint: remember to prove that $\sqrt{I}$ is an ideal, which is prime.)

4. Suppose that R is a commutative ring with 1 such that for every $x \in R$, there is some natural number $n > 1$ such that $x^n = x$. Show that every prime ideal of R is maximal.

5. Compute the Galois group of $x^4 - x^2 - 6$ over $\mathbb{Q}$.

6. Suppose V is a finite dimensional vector space over a field k and suppose that $A : V \to V$ is a k-linear endomorphism whose minimal polynomial is *not* equal to its characteristic polynomial. Show that there exist k-linear endomorphisms $B, C : V \to V$ such that $AB = BA$, $AC = CA$, but $BC \neq CB$

7. Suppose that K is an extension of $\mathbb{Q}$ of degree n. Let $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{C}$ be the distinct embeddings of K into $\mathbb{C}$. Let $\alpha \in K$. Regarding K as a vector space over $\mathbb{Q}$, let $\phi : K \to K$ be the linear transformation $\phi(x) = \alpha x$. Show that the eigenvalues of ϕ are $\sigma_1(\alpha), \ldots, \sigma_n(\alpha)$.

8. An R-module M is called *irreducible* if $M \neq 0$ and the only submodules of M are 0 and M. Now suppose that R is a commutative ring with 1 and that M is a left R-module. Show that M is irreducible if and only if M is isomorphic to R/I for a maximal ideal I of R.