Numerical Analysis Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Find a Gauss quadrature formula that approximates $\int_0^h f(x)dx \approx c_1 f(0)+c_2 f(c_3)$, where c_1,c_2 and $c_3 \in (0,h]$ are to be determined. Estimate the order of accuracy (in terms of $O(h^r)$) of this approximation.

2. Let $\{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_6\}$ be the vertices and edge midpoints of a triangle in counterclockwise direction, beginning with a vertex. Let P^2 be the linear space of polynomials of degree up to two. One can define a set of Lagrangian basis functions of P^2 associated with these points. Find the Lagrangian basis functions of P^2 associated with $\mathbf{A}_1$ and $\mathbf{A}_2$.

3. Consider an initial value problem $x' = f(x)$ with initial condition $x(0) = x_0$, where f is smooth enough function. What is the order of accuracy for the following scheme:
$$x_{k+1} = x_k + \frac{h}{2}(3f_k - f_{k-1})?$$
Find the interval for h that the scheme is absolutely stable. If the scheme is not stable, explain why.

4. Consider the scheme
$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \theta \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} + (1-\theta)\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2},$$
$0 \le \theta \le 1$. Analyze its stability for various θ.

5. Consider the Poisson equation $-\triangle u = f$ defined on a polygonal domain Ω with Dirichlet boundary condition $u|_{\partial\Omega} = g$. Let Ω be partitioned with a triangular mesh (without "hanging nodes"). Describe a piecewise linear continuous finite element method for solving this problem on the mesh and prove the existence and uniqueness of the numerical solution of the method.

6. Consider a symmetric matrix A given by
$$A = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0 & \cdots & 0 \\ 0.1 & 2 & 0.1 & 0 & 0 & \cdots & 0 \\ 0 & 0.1 & 3 & 0.1 & 0 & \cdots & 0 \\ 0 & 0 & 0.1 & 4 & 0.1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0.1 & 100 \end{bmatrix}.$$
Let $\vec{x}_0 = \frac{1}{10}[1, \cdots, 1]^T \in \mathbb{R}^{100}$ and $\mu = 50.25$. One computes $\vec{x}_n$ and λ_n by the following procedure,
$$\begin{aligned} &for \quad n = 1, 2, \cdots; \\ &\qquad \vec{w} = (A - \mu I)^{-1}\vec{x}_{n-1}; \\ &\qquad \vec{x}_n = \vec{w}/\|\vec{w}\|_2; \\ &\qquad \lambda_n = \vec{x}_n^T A \vec{x}_n; \\ &end; \end{aligned}$$

Are $\vec{x}_n$ and λ_n convergent? If so, what are they converge to and what are the rate of convergence? If not, explain why.

7. Consider to solve a linear system of equation $A\vec{x} = \vec{b}$ by iterative methods, where A is a $m \times m$ real symmetric positive definite matrix, denote its solution as $\vec{x}^*$. Design a numerical algorithm that finds $\vec{x}_n \in \vec{x}_{n-1} + \text{span}\{\vec{r}\}$, where $\vec{r} = \vec{b} - A\vec{x}_{n-1}$, such that the new residual at $\vec{x}_n$ is orthogonal to $\vec{r}$. Prove that $\vec{x}_n$ is the minimizer of

$$\min_{\vec{x} \in \vec{x}_{n-1} + \text{span}\{\vec{r}\}} [(\vec{x}^* - \vec{x})^T A(\vec{x}^* - \vec{x})]^{\frac{1}{2}}.$$