Numerical Analysis Comprehensive Exam
August 24, 2016

Student Number: ☐

Instructions: Complete up to 5 of the 8 problems (**3 must be from the last 4 problems**), and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider the ODE system $\begin{cases} q' = p \\ p' = -q \end{cases}$ and a one-step numerical integrator given by $\begin{cases} q_{k+1} = q_k + hp_k \\ p_{k+1} = p_k - hq_{k+1} \end{cases}$.

   1. What is the maximum value of h such that the method is stable for arbitrarily long time?
   2. Compare with the long time stability of $\begin{cases} q_{k+1} = q_k + hp_k \\ p_{k+1} = p_k - hq_k \end{cases}$; which method is better?

2. Consider three points Legendre-Gauss integration formula

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) + \frac{1}{525} f^{(4)}(\xi) \qquad (1)$$

   for some point $\xi \in (-1, 1)$. Derive nodes and weights for three points Legendre-Gauss quadrature for a function defined over the interval $[a, b]$, as well as an error expression similar to (1).

3. Approximate x^2 in $L^2(0,1)$ by a combination of $1, x$ and by a combination of x^3, x^4. Which approximation gives a smaller approximation error?

4. Consider $x' = f(x)$ with smooth enough f and one-step method given by

$$x_{k+1} = x_k + hf\left(\frac{x_{k+1} + x_k}{2}\right).$$

   Show that the method is 2nd-order.

5. Consider the initial value problem given by

$$u_t = 2u_{xx} + \sin 2t \cos 2x, \quad u|_{t=0} = u_0(x)$$

   Design an one step numerical scheme that is second order accurate in both space and time, and unconditionally stable. You must justify your answer.

6. Consider the Burgers' equation with a given initial condition,

$$u_t + uu_x = 0, \quad u|_{t=0} = u_0(x),$$

where $u_0(x)$ is non-negative. One uses the following upwind scheme to compute its numerical solution

$$\frac{U_j^{n+1} - U_j^n}{k} + U_j^n \frac{U_j^n - U_{j-1}^n}{h} = 0$$

where U_j^n is the numerical solution, k and h are time and space step sizes respectively. Does this scheme converge to the weak solution of the Burgers' equation? If yes, give your justification and find its accuracy. If not, explain why and give a modification so that it is convergent.

7. Consider a linear system of equations

$$A\vec{x} = \vec{f},$$

where $\vec{f} \in \mathbb{R}^n$ is given, A is a 5-band symmetric $n \times n$ matrix

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ c_1 & b_2 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & c_2 & b_3 & a_4 & b_4 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & c_{n-2} & b_{n-1} & a_n \end{bmatrix}.$$

Its entries a_i, b_i and c_i are random numbers following uniform distributions in intervals $[10, 20]$, $[0, 2]$ and $[-2, 0]$ respectively. Describe the Jacobi iteration to solve this system. Is the Jacobi iteration convergent? If your answer is yes, prove it. If not, explain why.

8. Suppose A is a 2010001×2010001 symmetric positive definite matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. One uses Gaussian Elimination with partial pivoting to solve the linear system $A\vec{x} = \vec{b}$. Give the best estimate for the relative error of the numerical solution, assuming the machine error is 10^{-16}. You must justify your answer.