Probability Comprehensive Exam
Aug 24, 2016

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Suppose (X_n) is a sequence of random vectors such that for some sigma-algebra $\mathcal{F}$, one has X_n and $\mathcal{F}$ independent for all n. If $X_n \to X$ almost surely, show that X and $\mathcal{F}$ are independent.

2. Let X be a random variable with continuous density function f and $f(0) > 0$. Let Y be a random variable with
$$Y = \begin{cases} \frac{1}{X} & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}.$$
and $Y_1, Y_2, \ldots$ be i.i.d. with distribution equal to that of Y. What is the value of the almost sure limit
$$\lim_n \frac{Y_1 + \cdots + Y_n}{n}?$$

3. Let $X_1, X_2, \ldots$ be i.i.d. with
$$\mathbb{P}(X_1 = 1) = 1/2 = \mathbb{P}(X_1 = -1).$$
Let C be the set of factorials:
$$C = \{k! : k \in \mathbb{N}\}.$$
Show that
$$\lim_n \mathbb{P}(X_1 + \cdots + X_n \in C) = 0.$$
Hint. You may want to start by covering part of $[0, \infty)$ by small intervals. Let $\epsilon > 0$ and $I > 0$, consider intervals $I_1, \ldots, I_I$, where $I_i = [(i-1)\epsilon, i\epsilon)$, and show that for large n, at most two of the sets $I_i \cap (C/\sqrt{n})$ are nonempty.

4. Let (X, Y) be a normal vector in $\mathbb{R}^2$ with mean zero and covariance matrix Σ, where
$$\Sigma = \begin{pmatrix} 5 & 1 \\ 1 & 10 \end{pmatrix}.$$
Find $\mathbb{E}X^2Y^2$.

5. Let ξ_1 and ξ_2 be independent random variables with characteristic functions $\varphi_1(u) = \frac{1-iu}{1+u^2}$ and $\varphi_2(u) = \frac{1+iu}{1+u^2}$ respectively. Find the probability that $\xi_1 + \xi_2$ takes values in $(3, +\infty)$.

6. Let $\{A_n\}$ be an infinite collection of independent events. Suppose that $\mathbb{P}(A_n) < 1$ for every $n \geq 1$. Show that $\mathbb{P}(A_n \, i.o.) = 1$ if and only if $\mathbb{P}(\cup A_n) = 1$.

7. Let X be a random variable taking values on the interval $[1,2]$. Find sharp lower and upper estimates on the quantity $\mathbb{E}X \cdot \mathbb{E}\frac{1}{X}$. Provide an example of a random variable for which the lower estimate is attained. Provide an example of a random variable for which the upper estimate is attained.

   Hint. For the upper bound, justify and use the inequality

$$ab \leq \frac{1}{2}\left(\frac{a}{2} + b\right)^2.$$

$$\mathbb{E}X \cdot \mathbb{E}\frac{1}{X} \leq \frac{1}{2}\left(\frac{1}{2}\mathbb{E}X + \mathbb{E}\frac{1}{X}\right)^2 = \frac{1}{2}\left(\mathbb{E}\left(\frac{X}{2} + \frac{1}{X}\right)\right)^2.$$

8. Show that for a sequence of random variables X_n, one has $X_n \to X$ in probability if and only if

$$\mathbb{E}\left[e^{\min\{2,|X_n - X|\}} - 1\right] \to 0,$$

   as $n \to \infty$.