Topology Comprehensive Exam
January 22, 2016

Student Number: ☐

Instructions: Complete up to 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X_1 and X_2 both be $S^1 \times S^1$. Fix $\theta_0 \in S^1$ and let X be obtained from $X_1 \cup X_2$ by identifying the point (θ_0, θ) in X_1 with (θ_0, θ) in X_2 for all $\theta \in S^1$. Use Van Kampen's Theorem to compute $\pi_1(X)$.

2. Let M be a 3 dimensional manifold. Suppose that α is a 1–form such that the 3–form $\alpha \wedge d\alpha$ is nowhere zero. Show that there is a unique vector field v such that $\alpha(v) = 1$ and $d\alpha(v, w) = 0$ for any other vector field w.

   You may use, without proof, the fact that if a vector space has a non-degenerate skew-symmetric bilinear pairing then it must have even dimension. Recall a non-degenerate skew-symmetric bilinear pairing is non-degenerate if the only vector that pairs to 0 with every other vector is the zero vector.

3. Let X be a connected, locally pathwise connected, and semi-locally simply connected topological space (so X has a universal cover). Prove that any connected n-fold covering space of X corresponds to a homomorphism of $\pi_1(X)$ into the symmetric group S_n whose image acts transitively on $\{1, \dots, n\}$, and conversely any such homomorphism is realized by a connected n-fold covering space of X.

4. Let $f : \mathbf{R}^5 \to \mathbf{R}^3$ be a smooth map. Show there is a round sphere S with center $(0, 0, 0)$ in $\mathbf{R}^3$ such that $f^{-1}(S)$ is a smooth (possibly empty) 4-dimensional submanifold of $\mathbf{R}^5$.

5. Let p, q be two distinct points of $\mathbf{C}$, the complex plane, and let X be a space obtained by attaching a cone to $\mathbf{C} \setminus \{p, q\}$ along a smoothly immersed circle α. Show that X has non-trivial fundamental group.

6. Let k be a positive integer and let $X_k = \mathbf{R}^3 \setminus Z_k$ where Z_k is the union of k distinct lines on $\mathbf{R}^3$ all passing through the origin.

   (a) Compute the fundamental group of X_k.

   (b) Show that any continuous map $S^2 \to X_k$ is homotopic to a constant map.

   (c) If $k > 1$ show that there is no retraction of X_1 onto X_k.

7. In this problem an *orientable manifold* is defined as a smooth manifold that admits an atlas whose transition functions have positive Jacobians.

   (a) Let $f : N \to O$ be a smooth immersion of n-dimensional manifolds such that O is orientable. Show that N is also orientable.

   (b) Show that for any non-orientable n-dimensional manifold N there is an orientable n-dimensional manifold O and a surjective immersion $O \to N$.

8. Fix points $v \in S^1$ and $w \in S^2$. Let U be an open subset of $S^1 \times S^2$ that contains $S^1 \times \{w\}$ and $\{v\} \times S^2$. Show that U is not diffeomorphic to an open subset of a simply-connected 3-dimensional manifold.